Real Analysis Comprehensive Examination–Math 921/922

Monday, June 1, 2009, 2:00-6:00p.m.

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Unless otherwise indicated $(X, \mathcal{M}, \mu)$ is a measure space.

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(1) Let $(p, q) \in (1, \infty)$ satisfying $1/p + 1/q = 1$ be given. Suppose that $\{\varphi_j\}_{j=1}^{\infty}$ is a dense subset of $L^q(X, \mu)$ with respect to its norm topology. Define $\omega : L^p(X, \mu) \to [0, \infty]$ by

$$\omega(f) := \sum_{j=1}^{\infty} \frac{1}{2^j \left(1 + \|\varphi_j\|_{L^q}\right)} \left| \int_X \varphi_j f \, \mathrm{d}x \right|.$$

a) Show that $\omega(f) < \infty$ for each $f \in L^p(X, \mu)$.
b) Verify that ω is a norm on $L^p(X, \mu)$.

(2) Let p, q, $\{\varphi_j\}_{j=1}^{\infty}$ and ω be given as in problem (1). Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^p(X, \mu)$ satisfies $\sup_{j \in \mathbb{N}} \|f_j\|_{L^p} < \infty$. Prove that $f_j \rightharpoonup 0$ in L^p if and only if $\lim_{j \to \infty} \omega(f_j) = 0$.

(3) Let ν be a measure on $\mathcal{M}$. Define $\nu_{\mathrm{s}} : \mathcal{M} \to [0, \infty]$ by

$$\nu_{\mathrm{s}}(E) := \sup\{\nu(F) : F \in \mathcal{M}, F \subseteq E \text{ and } \mu(F) = 0\}.$$

a) Prove that ν_{s} is a measure on $\mathcal{M}$ and that for each $E \in \mathcal{M}$ there is an $F \in \mathcal{M}$ that is admissible for $\nu_{\mathrm{s}}(E)$ such that $\nu_{\mathrm{s}}(E) = \nu(F)$.
b) Assume that ν_{s} is σ-finite. Prove that $\nu_{\mathrm{s}} \perp \mu$.

(4) With $\mathcal{L}$ denoting the Lebesgue measurable sets in $\mathbb{R}$ and m the Lebesgue measure on $\mathcal{L}$, define the signed measure $\nu : \mathcal{L} \to [-\infty, \infty)$ by $\nu(E) := 2m(E \cap [0, 1]) - m(E)$.
a) Provide, with justification, a Hahn decomposition of $\mathbb{R}$ with respect to ν.
b) Provide, with justification, a Jordan decomposition of ν.
c) Compute $\mathrm{d}\nu/\mathrm{d}m$.

(5) Let $(Y, \mathcal{N}, \nu)$ be a measure space. Suppose that $f \in L^+(\mu)$ and $g \in L^+(\nu)$ are $\mathbb{R}$-valued and satisfy

$$\mu(\{x \in X : f(x) \geq \alpha\}) = \nu(\{y \in Y : g(y) \geq \alpha\}) \quad \text{for each } \alpha \in [0, \infty).$$

Let $h : \mathbb{R} \to [0, \infty)$ be a non-decreasing continuously differentiable function such that $h(0) = 0$.
a) Verify that $h \circ f$ and $h \circ g$ are both measurable.
b) Prove that

$$\int_X h \circ f \, \mathrm{d}\mu = \int_Y h \circ g \, \mathrm{d}\nu.$$

(6) Let X be a nonempty set, and set

$$\mathcal{A} := \{E \subset X : E \text{ or } E^c \text{ is a finite set of elements}\}.$$

a) Prove that $\mathcal{A}$ is a σ-algebra on X if and only if X is a finite set of elements.
b) Determine, with justification, what the σ-algebra generated by $\mathcal{A}$ is; i.e. provide a characterization of the members of this σ-algebra.

(7) Let X be a nonempty set, let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, fix $x_0 \in X$, and define $\mu_0 : \mathcal{A} \to \{0, 1\}$ by

$$\mu_0(A) := \begin{cases} 0, & \text{if } x_0 \in A^c; \\ 1, & \text{if } x_0 \in A. \end{cases}$$

a) Show that μ_0 is a premeasure on $\mathcal{A}$.
b) Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the outermeasure induced by μ_0. Given $E \subseteq X$, prove that if $x_0 \in E$, then $\mu^*(E) = 1$.
c) Explain why the sets in $\mathcal{A}$ must be μ^*-measurable.

(8) Suppose that μ is a finite measure. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued functions. Prove that $f_j \to 0$ in measure if and only if

$$\lim_{j \to \infty} \int_X \frac{|f_j|}{1 + |f_j|} \, \mathrm{d}\mu = 0.$$