Real Analysis Comprehensive Examination

Monday, June 2, 2008, 2:00 – 6:00pm, 110 Avery Hall

- Work 6 out of 8 problems.
- Each question is worth 20 points.
- Write on one side of the paper only.

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1. (a) Let X be a set. Define the terms **outer measure** and **measurable set** on X. Suppose μ^* is an outer measure on X and $\mu^*(A) = 0$. Prove that A is measurable.

   (b) An outer measure is called **regular** if, given $E \subseteq X$ and $\epsilon > 0$, there is a measurable set A with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$. Prove that if μ^* is the outer measure induced by a measure μ on an algebra, then it is regular.

   (c) Construct an example of an outer measure that is not regular. (You can do this using only a finite set.)

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and f a non-negative measurable function on X. Let

$$G := \{(x, y) \in X \times \mathbb{R} \ : \ 0 \leq y \leq f(x)\}.$$

   Prove that G is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$ measurable and that

$$\mu \times m(G) = \int f \, d\mu.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Say that $E \subseteq X$ is **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. Let $\overline{\mathcal{M}}$ be the set of all locally measurable sets. Prove that $\overline{\mathcal{M}}$ is a σ-algebra. Define $\overline{\mu}$ on $\overline{\mathcal{M}}$ as

$$\overline{\mu}(E) := \sup\{\mu(A) \ : \ A \in \mathcal{M}, A \subseteq E\}.$$

   Show that when μ is semifinite, $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$.

4. Let $X = [0, 1]$ with Lebesgue measure and let $f_n(X) := \cos 2\pi n x$. Does $f_n \to 0$ almost everywhere? Does $f_n \to 0$ in measure? For fixed $1 < p < \infty$, prove that $f_n \to 0$ weakly in $L^p(X)$. Does this also hold for $p = 1$?

5. Let

$$F(x) := \begin{cases} 1 & ; \quad x < -1 \\ 1 - x^2 & ; \quad -1 \leq x \leq 1 \\ 1 & ; \quad x > 1 \end{cases}$$

   let μ_F be the signed Lebesgue-Stieltjes measure induced by F and let $d\mu_F = d\lambda + f dm$ (where m is Lebesgue measure). Answer each of the following questions, justifying your answer in terms of the appropriate definitions or theorems:

   (a) Give a formula for f and for λ.

   (b) Give two different Hahn decompositions of μ_F.

   (c) Give the Jordan decomosition of μ_F.

   (d) Give a formula for $|\mu_F|$.

6. (a) State Fatou's Theorem and the Dominated Convergence Theorem. Let $f \in L^1(\mathbb{R}, m)$ and prove $F(x) := \int_{[-\infty, x]} f(x) \, dm(x)$ is continuous.

   (b) Prove that if f is increasing on $[a, b]$ then

$$\int_{[a,b]} f'(x) \, dm(x) \leq f(b) - f(a)$$

7. A set $S \subseteq \mathbb{R}$ is called **measure dense** if $m(S \cap (a, b)) > 0$ for all $a < b$. Find a set $A \subseteq \mathbb{R}$ such that both A and A^c are measure dense.

8. Let $f_n, f \in L^p$ and suppose $\|f_n - f\| \to 0$. Prove that there is a subsequence f_{n_k} which converges to f pointwise almost everywhere. Give an example to show that it is possible that the original sequence f_n does not converge to f almost everywhere.