Real Analysis Comprehensive Examination–Math 921/922
Wednesday, May 30, 2007, 2:00-6:00p.m., 111 Avery Hall

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

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(1) Consider the family of sets $\mathcal{A}$ consisting of all $E \subseteq [0,1]$ such that χ_E is Riemann integrable in $[0,1]$.
a) Show that $\mathcal{A}$ is an algebra of sets in $[0,1]$. (You may assume, without proof, that the product of two Riemann integrable functions is also Riemann integrable.)
b) Show that $\mathcal{A}$ is not a σ-algebra.
c) Prove or disprove the following statement: the σ-algebra generated by $\mathcal{A}$ contains $\mathcal{B}_{[0,1]}$.

(2) Let $\alpha \in (1,\infty)$ be given. Suppose that μ is the measure on $\mathcal{L}$, the Lebesgue measurable sets in $\mathbb{R}$, given by $\mu(E) := \int_E (1+|x|^\alpha)^{-1}\,dx$ for all $E \in \mathcal{L}$.
a) Prove that the function $x \mapsto |x|^\beta$ is Borel-measurable for each $\beta \in \mathbb{R}$.
b) Given $p \in [1,\infty]$, for which values of $\beta \in \mathbb{R}$ is the function $x \mapsto |x|^\beta$ in $L^p(\mu)$.

(3) Given $f \in L^1(m)$, for each $t \in (0,\infty)$, define the **Laplace transform of f at t**, denoted by $\mathscr{F}(t)$, by $\mathscr{F}(t) := \int_{(0,\infty)} e^{-tx} f(x)\,dx$.
a) Show that $t \mapsto \mathscr{F}(t)$ is continuous on $(0,\infty)$.
b) Is $t \mapsto \mathscr{F}(t)$ differentiable for $t \in (0,\infty)$? Justify your answer.

(4) Let μ be the counting measure on $\mathcal{P}(\mathbb{N})$ and m be the Lebesgue measure restricted to the Lebesgue measurable sets in $[0,1]$, denoted by $\mathcal{L}_{[0,1]}$. Define the function $f : \mathbb{N}\times[0,1] \to [0,\infty)$ by $f(k,x) := ke^{-kx}\cdot 2^{-k}$. Show that f is $(\mathcal{P}(\mathbb{N})\otimes\mathcal{L}_{[0,1]}, \mathcal{B}_{[0,\infty)})$-measurable, and compute $\int_{\mathbb{N}\times[0,1]} f\,d(\mu\times m)$. Justify all your steps.

(5) Let $p \in [1,\infty)$ be given. Suppose that $f \in L^p(m)$, where m is the Lebesgue measure restricted to the Borel sets $\mathcal{B}_{\mathbb{R}^n}$. Prove that $\lim_{h\to 0}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx = 0$. (Hint: First show that this is true if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous with compact support.)

(6) Let $(X,\mathcal{M},\mu)$ be a finite measure space and $p,q \in (1,\infty)$. Suppose that $\{f_k\}_{k=1}^\infty \subset L^1(\mu)$ and $\{g_k\}_{k=1}^\infty \subset L^p(\mu)$ satisfy
- there is an $f \in L^\infty(\mu)$ such that $f_k \to f$ a.e.;
- there is a $g \in L^p(\mu)$ such that $g_k \rightharpoonup g$ in L^p (weak convergence);
- $\sup_{k\in\mathbb{N}} \|f_k g_k\|_{L^q} < +\infty$ and $\sup_{k\in\mathbb{N}} \|g_k\|_{L^p} < +\infty$.

Prove that $f_k g_k \rightharpoonup fg$ in L^q. (You may assume, without proof, that $fg \in L^q(\mu)$.)

(7) Let X be a nonempty set and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. Define the **contracted outer measure** $\mu_c^* : \mathcal{P}(X) \to [0,\infty]$ by

$$\mu_c^*(E) := \sup\{\mu^*(F) : F \subseteq E \text{ and } \mu^*(F) < +\infty\}.$$

a) Prove that μ_c^* is an outer measure on X.
b) Prove that $E \subseteq X$ is μ_c^*-measurable if and only if E is μ^*-measurable.

(8) Let $\{r_k\}_{k=1}^\infty$ be an enumeration of the rational numbers in $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{\{n\in\mathbb{N}\,:\, r_n \le x\}} 2^{-n}.$$

a) Prove that f is continuous at each irrational number but discontinuous at each rational number.
b) Prove that f is everywhere right-continuous.
c) Show that f' exists a.e. and $f' = 0$ a.e.. (Hint: consider the Lebesgue-Stieltjes measure μ_f.)