Real Analysis Comprehensive Examination–Math 921/922
Thursday, June 1, 2006, 2:00-6:00p.m., 110 Avery Hall

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Throughout the exam, the Lebesgue measure is denoted by m and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

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(1) Let $E \subset [0,1]$ be a Lebesgue measurable set with $m(E) > 0$, and let $0 < \epsilon < 1$ be given.
a) (12 points) Prove that there exists an open interval I such that $m(E \cap I) > \epsilon\, m(I)$.
b) (8 points) Assume $A \subset [0,1]$ is a Lebesgue measurable set such that $m(A \cap I) \geq \frac{1}{5} m(I)$ for every open interval $I \subset [0,1]$. Prove that $m(A) = 1$.

(2) Show all technical details in evaluating: $\displaystyle\lim_{n\to\infty} \int_{[0,\infty)} \frac{n}{1+n^2x^2}\, dm.$

(3) Consider the measure spaces $([0,1], \mathcal{B}_{[0,1]}, m)$ and $([0,1], \mathcal{B}_{[0,1]}, \nu)$, where ν is the counting measure. Let $D = \{(x,x) :\ x \in [0,1]\}$ be the diagonal in $[0,1] \times [0,1]$. Show the integrals $\int_{[0,1]} \int_{[0,1]} \chi_D dm d\nu$, $\int_{[0,1]} \int_{[0,1]} \chi_D d\nu dm$, $\int_{[0,1]\times[0,1]} \chi_D d(m \times \nu)$ are all unequal; and explain why this does not contradict Tonelli's Theorem.

(4) Consider the functions $f : [a,b] \to [c,d]$ and $g : [c,d] \to \mathbb{R}$.
a) (8 points) Prove that: if f and g are absolutely continuous and f is strictly increasing; then $g \circ f$ is absolutely continuous on $[a,b]$.
b) (12 points) Give an example of absolutely continuous functions f and g on $[0,1]$ such that $g \circ f$ is not absolutely continuous. (Hint: recall that if a function h is not of bounded variation on $[0,1]$ then h is not absolutely continuous).

(5) Let $\{r_j\}_{j=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$, the rationals in $[0,1]$. For each Lebesgue measurable set E in $[0,1]$, let $\nu(E) := \displaystyle\sum_{\{j:\ r_j \in E\}} \frac{(-1)^j}{2^j}$. Then, ν is a signed measure on $([0,1], \mathcal{L}_{[0,1]})$.
You need **not** prove this fact.
a) (8 points) Find a Hahn decomposition for ν.
b) (7 points) Find the Jordan decomposition of ν.
c) (5 points) Is $|\nu|$ a complete measure? Justify your answer.

(6) Let $f \in L^2([0,1], m)$ be $\mathbb{R}$-valued function such that $\int_{[0,1]} t^{2n+1} f(t) dm(t) = 0$, for $n = 0, 1, 2, \ldots$. Prove that $f = 0$ a.e. $[0,1]$. (Hint: First, consider the change of variables $x = t^2$ in the above formula).

(7) Let $(X, \mathcal{M}, \mu)$ be a **finite** measure space and $f_n \in L^p(X, \mathcal{M}, \mu)$; $n = 0, 1, 2\ldots$, be $\mathbb{R}$-valued functions, where $1 < p < \infty$. Assume that $\|f_n\|_p \leq 3$ for all $n = 0, 1, 2, 3\ldots$ and $f_n \to f_0$ pointwise on X.
a) (15 points) Prove that $f_n \to f_0$ weakly in $L^p(X, \mathcal{M}, \mu)$.
b) (5 points) Give an example in $\mathbb{R}$ that shows the result in part a) can fail if $p = 1$.

(8) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued continuous functions on a complete metric space X. Use the Baire Category Theorem to prove the following special case of the Uniform Boundedness Principle: *If for every $x \in X$, $M_x := \sup_{n\in\mathbb{N}} |f_n(x)| < \infty$, then there exist a nonempty open set $U \subseteq X$ and a constant $C > 0$ such that $|f_n(x)| \leq C$ for all $x \in U$ and all $n \in \mathbb{N}$.*