Your Exam ID Code: ____________________

Real Analysis Comprehensive Examination

Tuesday, May 31, 2005, 1:00–5:00p.m., Avery 119

• Work 6 of the 8 problems below. • Each problem is worth 20 points. • Write on one side of the paper only.

1) Define $\mathcal{S}$ to be the collection of all subsets $E \subseteq \mathbb{R}$ such that E is countable or E^c is countable. For $E \in \mathcal{S}$, define $\mu(E) = 1$ if E is uncountable and $\mu(E) = 0$ if E is countable. Finally, let $f(x) = x\chi_{\mathbb{Q}}(x)$.

a) (5 points) Show that $\mathcal{S}$ is a σ-algebra.

b) (5 points) Prove that μ is a measure.

c) (5 points) Prove that f is a $\mathcal{S}$-measurable function.

d) (5 points) Prove f is μ-integrable and find $\int_{\mathbb{R}} f\,d\mu$.

2) Let $\mathcal{S} := \mathcal{P}(\mathbb{N})$ be the collection of all subsets of the natural numbers. Given a function $h : \mathbb{N} \to [0, \infty]$, let μ be the measure on $(\mathbb{N}, \mathcal{S})$ defined by $\mu(E) = \sum_{x \in E} h(x)$. If $f : \mathbb{N} \to \mathbb{C}$ is μ-integrable, give (and completely justify) a formula for $\int_{\mathbb{N}} f\,d\mu$ (in terms of h and f).

3) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that the graph, $G := \{(x, f(x)) : x \in [0, 1]\}$ is $m \times m$-measurable and satisfies $(m \times m)(G) = 0$, where $m \times m$ is two-dimensional Lebesgue measure (i.e. the product measure of m with itself).

4) Let $(X, \mathcal{S}, \mu)$ be a measure space, $1 < p < \infty$, and let q be the conjugate exponent, i.e. $p^{-1} + q^{-1} = 1$. Let $f, f_n \in L^p(X, \mu)$ and $g, g_n \in L^q(X, \mu)$. Prove the following (unrelated) statements.

a) (7 points) If $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_q \to 0$, then $\|f_n g_n - fg\|_1 \to 0$.

b) (13 points) If $\|f_n - f\|_p \to 0$, then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that for almost every $x \in X$, $f_{n_k}(x) \to f(x)$.

5) Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. As usual, let f^+ be the positive part of f, i.e. $f^+(x) = \max\{0, f(x)\}$. Remarkably, $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln(1 + e^{nf(x)})\,dx = \int_0^1 f^+(x)\,dx$. Give a complete and detailed proof of this. (You may use without proof the fact that for $t \in (0, \infty)$, $\ln(1 + e^t) < t + \ln 2$.)

6) For each $n \in \mathbb{N}$, let $I_n = [0, 1]$, and let $X = \prod_{n\in\mathbb{N}} I_n$ be the product space, equipped with the product topology.

a) (5 points) Explain why a function $f : X \to \mathbb{R}$ may be regarded as a function of countably many variables $x_1, x_2, x_3, \ldots$.

b) (15 points) For any finite subset $F \subseteq \mathbb{N}$, let

$$\mathcal{C}_F := \{g : X \to \mathbb{R} \,|\, g \text{ is continuous and depends only on the the finitely many variables } \{x_i\}_{i\in F}\}.$$

Show that if $f : X \to \mathbb{R}$ is continuous and $\varepsilon > 0$ then there exists a finite subset $F \subseteq \mathbb{N}$ and $h \in \mathcal{C}_F$ such that for every $x \in X$, $|f(x) - h(x)| < \varepsilon$.

7) For any $f : \mathbb{R} \to \mathbb{R}$, let $Z_f := \{x \in \mathbb{R} : f(x) = 0\}$. Define

$$D := \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is bounded and there exists } F : \mathbb{R} \to \mathbb{R} \text{ such that } F'(x) = f(x)\, \forall x \in \mathbb{R}\} \quad \text{and}$$
$$D_0 := \{f \in D : Z_f \text{ is dense in } \mathbb{R}\}.$$

a) (10 points) Prove that if $f \in D$ then Z_f is a G_δ set.

b) (10 points) It follows from a theorem in elementary analysis that if $f_n \in D_0$ and f_n converges uniformly to f, then $f \in D$. Prove however, that actually $f \in D_0$. (You may use the fact that $f \in D$ without proof. Also, you might find a result of Baire useful.)

8) Suppose $F : \mathbb{R} \to \mathbb{R}$ is an increasing function. Given $a, b \in \mathbb{R}$ with $a < b$, prove that $F(b) - F(a) \geq \int_a^b F'(x)\,dx$.