Your Exam ID Code: ____________________

Real Analysis Comprehensive Examination–Math 921/922
Friday, June 4, 2004, 1:00-5:00p.m., 208 Oldfather Hall

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Throughout the exam, the Lebesgue measure is denoted by m and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

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(1) Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$. Let $f : \mathbb{R} \longrightarrow [0, \infty]$ and assume that $f \in L^1(\mathbb{R}, \mu)$.
a) Prove that the set $\{x \in \mathbb{R} : \ f(x) > 0\}$ is σ-finite.
b) Assume there exists a compact subset K of $\mathbb{R}$ such that $\mu(K) = \infty$. Prove that: for every $\delta > 0$ there exists an interval $[a, b]$ such that $b - a < \delta$ and $\mu[a, b] = \infty$.

(2) Let $f \in L^1(X, \mathcal{M}, \mu)$ be a non-negative function satisfying $\int_X (f)^n d\mu = \int_X f d\mu$ for all $n \in \mathbb{N}$. Prove that f must be μ-almost everywhere equal to a characteristic function χ_E for some set $E \in \mathcal{M}$.

(3) a) Let $f : \mathbb{R} \longrightarrow [0, \infty)$ be Borel measurable and assume that $f \in L^1(\mathbb{R}, m)$. Define $\phi : (0, \infty) \longrightarrow [0, \infty)$ by $\phi(y) = m\{x \in \mathbb{R} : \ f(x) > y\}$. Prove that: $\int_{(0,\infty)} \phi(y) dm(y) = \int_{\mathbb{R}} f(x) dm(x)$. (Hint: Let $G = \{(x, y) \in \mathbb{R} \times (0, \infty) : \ f(x) - y > 0\}$. Show that G is $\mathcal{B}_{\mathbb{R}^2}$–measurable and hence so is χ_G).

(4) Show all technical details in evaluating: $\displaystyle\lim_{n\to\infty} \int_{[2,\infty)} \frac{n \sin(x/n)}{x^2 (\log x)^2} \, dm(x)$.

(5) Let $\mathcal{A}$ be any non-empty set and consider the measure space $l^p(\mathcal{A}) := L^p(\mathcal{A}, \mathcal{P}(\mathcal{A}), \mu)$ where μ is the counting measure and $1 < p < \infty$. Assume that $\{f_n\} \subset l^p(\mathcal{A})$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ and $f_n \longrightarrow f$ point-wise on $\mathcal{A}$. Prove that: $f_n \longrightarrow f$ weakly in $l^p(\mathcal{A})$.

(6) Let $f_n : [0, 1] \longrightarrow \mathbb{R}$ be Lebesgue measurable functions for $n \in \mathbb{N}$. Prove or disprove each of the following:
a) $f_n \longrightarrow f$ a.e-$[0, 1]$ and $|f_n(x)| \leq \frac{1}{x^{1/4}}, \ x \in (0, 1] \implies \|f_n - f\|_2 \longrightarrow 0$.
b) $f_n \longrightarrow f$ in $L^p([0, 1], m)$ where $p > 2 \implies f_n \longrightarrow f$ in $L^2([0, 1], m)$.
c) $\int_{[0,1]} |f_n - f|^{1/2} \, dm \longrightarrow 0 \implies f_n \longrightarrow f$ in measure.

(7) Consider the measurable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$. The finite signed measure ν is defined on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ as follows: For $E \in \mathcal{B}_{\mathbb{R}}$

$$\nu(E) = \int_{E\cap(-\infty,0]} e^x dm(x) - \int_{E\cap[0,\infty)} e^{-x} dm(x) + a \, \delta_1(E);$$

where δ_1 denotes the unit mass at $x = 1$ and $a \geq 0$. With justification:
a) Give the Hahn Decomposition of $\mathbb{R}$ with respect to ν when $a > 0$.
b) Give the Jordan Decomposition of ν when $a > 0$.
c) Identify $|\nu|$, the total variation of ν and find its Lebesgue Decomposition with respect to the Lebesgue measure m when $a > 0$.
d) If $a = 0$, identify $\frac{d\nu}{dm}$, the Radon-Nikodym derivative of ν with respect to m.

(8) Let P be the set of all irrational numbers, viewed as a metric space with the usual Euclidian metric. For each $n \in \mathbb{N}$, let $G_n \subseteq P$ be a (relatively) open subset of P which is also dense in P. Prove that $\bigcap_{n=1}^{\infty} G_n$ is dense in P.