Your Exam ID Code: ____________________

Real Analysis Comprehensive Examination

Thursday, June 13, 2002, 1:00–5:00p.m., 304 Oldfather

• Work 6 of the 8 problems below. • Each problem is worth 20 points. • Write on one side of the paper only.

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1) Give the definition of an outer measure on a set X. If $E \subseteq \mathbb{R}$, give the definition of $m^*(E)$ and, using only the relevant definitions, prove that m^* is an outer measure on $\mathbb{R}$.

2) Carefully prove that $L^1(\mathbb{R}, m)$ is separable.

3) Suppose that $G : \mathbb{R} \to \mathbb{R}$ is continuous and increasing. Fix $a < b$ and put $c = G(a)$, $d = G(b)$. Prove that if $f : [c, d] \to \mathbb{C}$ is Borel measurable and integrable on $[c, d]$, then

$$\int_c^d f(y)\, dy = \int_a^b f(G(x)) d\mu_G(x).$$

(Recall that μ_G is the unique Borel measure on $\mathbb{R}$ such that $\mu_G((x, y]) = G(y) - G(x)$.)

4) This question has three parts—work them in any order you wish. Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are Lebesgue integrable, and define $H(x, t) = f(x)g(t - x)$. Show the following statements hold.

a) (6 points) H is measurable (with respect to the product of Lebesgue measure with itself).

b) (7 points) For almost every $t \in \mathbb{R}$, the integral $\int_{\mathbb{R}} f(x)g(t - x)\, dx =: (f * g)(t)$ exists.

c) (7 points) The function $f * g$ is integrable and $\int_{\mathbb{R}} |(f * g)(t)|\, dt \le \left(\int_{\mathbb{R}} |f(t)|\, dt \right) \left(\int_{\mathbb{R}} |g(t)|\, dt \right)$.

5) For $b > 0$, put $F(b) = \int_0^\infty \frac{\cos b + y \sin b}{1 + y^2} e^{-by}\, dy$. Find $\lim_{b\to\infty} F(b)$. Give a detailed proof that your answer is correct.

6) Let A be a non-empty set and let $X = [0, 1]^A$ be the set of all functions from A into $[0, 1]$. Equip X with the product topology. For $\alpha \in A$, let $\pi_\alpha : X \to \mathbb{R}$ be defined by $\pi_\alpha(f) = f(\alpha)$. If $\mathfrak{A}$ is the algebra generated by $\{\pi_\alpha : \alpha \in A\}$ and the constant functions, show that given $H \in C(X)$ and $\varepsilon > 0$, there exists $K \in \mathfrak{A}$ such that $|H(f) - K(f)| < \varepsilon$ for every $f \in X$.

7) Suppose that for $i = 1, 2$, $F_i : [0, 1] \to \mathbb{R}$ is absolutely continuous.

a) (10 points) Prove that F_1F_2 is absolutely continuous on $[0, 1]$.

b) (10 points) Consider the equation, $\int_0^1 (F_1(t)F_2'(t) + F_1'(t)F_2(t))\, dt = F_1(1)F_2(1) - F_1(0)F_2(0)$. Justify the validity of this equation, being sure to make clear why the integral exists and the equality holds.

8) Suppose that $\{F_n\}_{n=1}^\infty$ is a pairwise disjoint sequence of closed subsets of $[0, 1]$ such that $[0, 1] = \bigcup_{n=1}^\infty F_n$.

a) (13 points) Let $G_k := \bigcup_{n=k}^\infty F_n$ and set $X := \bigcap_{k=1}^\infty \overline{G_k}$. Prove that $X = \emptyset$. (*Suggestion:* Consider the sets $U_k = X \cap G_k$. Supposing $X \neq \emptyset$, apply a theorem of Baire to make a conclusion about $\cap_{k=1}^\infty U_k$.)

b) (7 points) Prove that there exists $N \in \mathbb{N}$ such that whenever $n \ge N$, $F_n = \emptyset$.