Real Analysis Comprehensive Examination–Math 921/922
Friday, January 21, 2011, 2:00-6:00 p.m.

- Work 6 out of 8 problems.
- Each problem is worth 20 points.
- Write on one side of the paper only and hand your work in order.
- Throughout the exam, the Lebesgue measure is denoted by \( m \), \( B_\mathbb{R} \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R} \), and \( (X, \mathcal{M}, \mu) \) denotes a general measure space.

1. Let \( E \subset \mathbb{R} \) be a Lebesgue measurable set such that \( m(E) = \infty \). Prove that:
   a) \( E = V \setminus N_1 \), where \( V \) is a \( \mathcal{G}_2 \) set and \( N_1 \) is a null set for \( m \).
   b) \( E = H \cup N_2 \), where \( H \) is an \( \mathcal{F}_\sigma \) set and \( N_2 \) is a null set for \( m \).

2. Let \( \{f_n\} \) and \( f \) be \( \mathcal{M} \)-measurable functions on \( (X, \mathcal{M}, \mu) \), \( n \in \mathbb{N} \), such that \( f_n \geq 0 \) and \( f_n \rightarrow f \) in measure. Prove that:
   \[ \int_X f \, d\mu \leq \lim \inf_{n \to \infty} \int_X f_n \, d\mu. \]
   (Hint: First show that every subsequence of \( \{f_n\} \) converges to \( f \) in measure.

3. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a Borel measurable function and let \( g_n(x) = (\cos f(x))^{2n} \), \( x \in [0, 1] \), \( n \in \mathbb{N} \).
   a) Carefully explain why \( g_n \) is Borel measurable for every \( n \in \mathbb{N} \).
   b) Show all technical details in evaluating:
   \[ \lim_{n \to \infty} \int_{[0, 1]} g_n(x) \, dm. \]

4. Let \( (X, \mathcal{M}, \mu) \) be a finite measure space. Assume that \( f_n, f : X \rightarrow \mathbb{R} \) are \( \mathcal{M} \)-measurable functions such that \( \|f_n\|_2 \) and \( \|f\|_2 \leq 1 \), \( n \in \mathbb{N} \); where as usual, \( \|f\|_2 := \int_X |f|^2 \, d\mu \). Further assume that:
   \[ f_n \rightarrow f \text{, } \mu\text{-a.e.} \]
   a) Prove that:
   \[ f_n \rightarrow f \text{ in } L^p(X, \mathcal{M}, \mu), \text{ for each } p \in [1, 2). \]
   (Hint: Egoroff’s Theorem).
   b) Provide a counterexample showing that the result in part a) fails if \( p = 2 \).

5. Let \( g \in L^\infty(\mathbb{R}, m) \) and \( p \in [1, \infty) \) be fixed. For \( f \in L^p(\mathbb{R}, m) \), define \( T(f)(x) = g(x)f(x) \).
   Prove that \( T : L^p(\mathbb{R}, m) \rightarrow L^p(\mathbb{R}, m) \) is a bounded linear map and that \( \|T\| = \|g\|_\infty \).
   (Don’t forget to address the case \( p = \infty \).

6. Let \( f \in L^1((0, 1), B_{(0,1)}, m) \) and \( g(x) := \int_{(x, 1)} \frac{1}{y} f(y) \, dm(y), x \in (0, 1) \).
   Prove that:
   \[ \int_{(0,1)} g(x) \, dm(x) = \int_{(0,1)} f(y) \, dm(y). \]

7. Let \( \{r_k\}_{k=1}^\infty \) be an enumeration of the rational numbers in \( \mathbb{R} \). Define \( F : \mathbb{R} \rightarrow \mathbb{R} \) by
   \[ F(x) := \sum_{n \in \mathbb{N} : r_n \leq x} 2^{-n}. \]
   a) Prove that \( F \) is continuous at each irrational number but discontinuous at each rational number.
   b) Prove that \( F \) is everywhere right-continuous.
   c) Show that \( F' \) exists a.e. and \( F'' = 0 \) a.e.. (Hint: consider the Lebesgue-Stieltjes measure \( \mu_F \).)

8. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a \( C^\infty \)-function on \([a, b] \) (i.e., \( f \) is infinitely differentiable on \([a, b] \)) with the property that:
   for every \( x \in [a, b] \) there exists an integer \( n_x \in \mathbb{N} \) such that \( f^{(n_x)}(x) = 0 \). Here, \( f^{(n_x)} \) stands for the derivative of order \( n_x \). Prove that \( f \) is a polynomial on \([a, b] \).
   (Suggestions: For each \( n \in \mathbb{N} \), put \( A_n = \{ x \in [a, b] : f^{(n)}(x) = 0 \} \) and \( E = \{ x \in [a, b] : \text{ there does not exist a neighborhood of } x \text{ on which } f \text{ is a polynomial} \} \).
   Use a Theorem of Baire on \( E \).