(1) a) Assume that $\mu$ is a finite measure. With $q \in (1, \infty)$, let $\{f_k\}_{k=1}^{\infty} \subset L^q(X, \mu)$ and $f \in L^q(X, \mu)$ be given. Suppose also that
\begin{itemize}
  \item $\sup_{k \in \mathbb{N}} \|f_k\|_{L^q} < \infty$ and
  \item $f_k(x) \to f(x)$ for a.e. $x \in X$.
\end{itemize}
Prove that $f_k \to f$ in $L^p$ for each $p \in [1, q)$.
b) Must the result in part a) remain true if $\mu$ is only assumed to be a $\sigma$-finite measure? (Justify your answer.)

(2) Assume that $\mu$ is a finite measure, and let $f : X \to [0, \infty)$ be an $\mathcal{M}$-measurable function. Suppose that $g : [0, \infty) \to [0, \infty)$ is an increasing function. Prove that
\[ \int_X g(f(x)) \, d\mu \geq \int_0^\infty g'(t) \mu(\{x \in X : f(x) > t\}) \, dt. \]
(You may assume without proof that the composition $g \circ f$ is $\mathcal{M}$-measurable.)

(3) Provide all technical details to evaluate $\lim_{k \to \infty} \int_{[0, \infty)} \frac{\cos(x/k)}{(1 + (x/k))^q} \, dx$.

(4) With $m$ the Lebesgue measure on the Lebesgue measurable sets $\mathcal{L}$ in $\mathbb{R}$, define the signed measure $\nu : \mathcal{L} \to \mathbb{R}$ by $\nu(E) := \int_E x(x - 1)e^{-x^2} \, dx$.
a) Provide a Hahn decomposition of $\mathbb{R}$ with respect to $\nu$.
b) Provide the Jordan decomposition of $\nu$.
c) Compute $\frac{d\nu}{dm}$.

(5) With $m$ the Lebesgue measure on the Borel sets in $\mathbb{R}^n$, let $f \in L^1(\mathbb{R}^n, m)$ be given. Recall the definition of the Hardy-Littlewood maximal function of $f$:
\[ Mf(x) := \sup_{r > 0} \frac{1}{m(B(r, x))} \int_{B(r, x)} |f(y)| \, dm(y). \]
a) Prove that, for each $t \geq 0$, the set $\{x \in \mathbb{R}^n : Mf(x) > t\}$ is open.
b) Argue that the function $Mf$ is Borel measurable.

(6) Let $X$ be a nonempty set, and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on $X$. Define $\nu^* : \mathcal{P}(X) \to [0, \infty]$ by
\[ \nu^*(E) := \inf \{\mu^*(F) : E \subseteq F \text{ and } F \text{ is } \mu^*\text{-measurable}\}. \]
a) Prove that $\nu^*$ is an outer measure on $X$.
b) Prove that if $E$ is $\mu^*$ measurable, then $E$ is $\nu^*$-measurable and $\mu^*(E) = \nu^*(E)$.

(7) Let $m$ be the Lebesgue measure on the Borel sets in $\mathbb{R}$. Define $\{f_k\}_{k=1}^{\infty} \subset L^\infty(\mathbb{R}, m)$ by $f_k(x) := k\chi_{[0,k]}(x)$. Justify your answers to the following questions.
a) Does $f_k \to 0$ in measure? 
b) Does $f_k(x) \to 0$ for a.e. $x \in \mathbb{R}$? 
c) Does $f_k \to 0$ in $L^1$? 
d) Does $f_k \to 0$ in $L^1$?

(8) Let $Y \subseteq X$ be given. Show that $\mathcal{N} := \{E \cap Y : E \in \mathcal{M}\}$ is a $\sigma$-algebra of subsets of $Y$. 

---

**Real Analysis Comprehensive Examination—Math 921/922**

Thursday, January 21, 2010, 2:00-6:00 p.m., 347 Avery Hall

- Work 6 out of 8 problems. 
- Each problem is worth 20 points. 
- Write on one side of the paper only and hand your work in order. 
- Unless otherwise indicated $(X, \mathcal{M}, \mu)$ is a measure space.