Real Analysis Comprehensive Examination–Math 921/922

Wednesday, January 23, 2008, 2:30-6:30p.m., 111 Avery Hall

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

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(1) Show all technical details in evaluating: $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n\sin(x/n)}{x+x^3}\,\mathrm{d}x$.

(2) Let $p, q \in [1,\infty]$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ be given. Suppose that $f \in L^p(m)\cap L^1(m)$ and $g \in L^q(m)\cap L^1(m)$ be given, where m is the Lebesgue measure on $\mathbb{R}^n$. For each $\mathbf{y}\in\mathbb{R}^n$, define $F(\mathbf{y}) := \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})g(\mathbf{x})\,\mathrm{d}\mathbf{x}$.
a) Show that the function $\mathbf{y}\mapsto F(\mathbf{y})$ is uniformly continuous on $\mathbb{R}^n$.
b) For which $r\in[1,\infty]$, if any, is $F\in L^r(m)$? Justify your answer.

(3) A function $f:[a,b]\to\mathbb{R}$ is said to be **singular** on $[a,b]$ if f' exists and is 0 for m-a.e. $x\in[a,b]$.
a) Prove or disprove: if $f:[a,b]\to\mathbb{R}$ is singular on $[a,b]$ then f is constant on $[a,b]$.
b) Show that if $f\in BV([a,b])$, then there are functions $f_a, f_s : [a,b]\to\mathbb{R}$ such that f_a is absolutely continuous on $[a,b]$, f_s is singular on $[a,b]$ and $f=f_a+f_s$ on $[a,b]$. Note that the decomposition is not unique. (Hint: use a Borel measure where f is a distribution function.)

(4) Let m denote the Lebesgue measure on $\mathbb{R}$. Given $f\in L^\infty(m)$, define $\{f_n\}_{n=1}^\infty \subset L^\infty(m)$ by $f_n(x) := f(nx)$, for each $n=1,2,\ldots$. Suppose, for each $E\in\mathcal{L}$ with finite measure, that

$$\lim_{n\to\infty}\int_E f_n(x)\,\mathrm{d}x = 0.$$

For each $p\in[1,\infty)$, prove that $f_n \rightharpoonup 0$ in $L^p(m)$ (weak convergence).

(5) Let X be a non-empty set. A **Dynkin system on** X is a family of sets $\mathcal{D}\subseteq\mathcal{P}(X)$ such that
- $X\in\mathcal{D}$;
- if $\{A_n\}_{n=1}^\infty\subseteq\mathcal{D}$ is a sequence of mutually disjoint sets, then $\cup_{n=1}^\infty A_n\in\mathcal{D}$;
- if $A,B\in\mathcal{D}$ with $A\subseteq B$, then $(B\backslash A)\in\mathcal{D}$.

a) Prove that a Dynkin system $\mathcal{D}$ on X is a σ-algebra if $A\cap B\in\mathcal{D}$ whenever $A,B\in\mathcal{D}$.
b) Let $\mathcal{E}\subseteq\mathcal{P}(X)$ be given. Show that there is a unique smallest Dynkin system $\mathcal{D}$ such that $\mathcal{E}\subseteq\mathcal{D}$. Call this system **the Dynkin system generated by** $\mathcal{E}$.
c) Suppose that $\mathcal{E}\subseteq\mathcal{P}(X)$ possesses the following property: $A\cap B\in\mathcal{E}$ whenever $A,B\in\mathcal{E}$. Prove that the σ-algebra generated by $\mathcal{E}$ equal to the Dynkin system generated by $\mathcal{E}$.

(6) Let X be a nonempty set and $\mu^*:\mathcal{P}(X)\to[0,\infty)$ be a finite outer measure. Assume that $\mu^*(X)>0$. Suppose that $A\subseteq X$ satisfies $\mu^*(A)>0$, and define $\mu_A^*:\mathcal{P}(X)\to[0,\infty)$ by

$$\mu_A^*(B) := \frac{\mu^*(A\cap B)}{\mu^*(A)} \quad \text{for each } B\subseteq X.$$

a) Verify that μ_A^* is an outer measure on X.
b) Show that if B is μ^*-measurable, then it is also μ_A^*-measurable.
c) Let $\mathcal{M}$ denote the set of all μ^*-measurable sets, and for each $A\subseteq X$ satisfying $\mu^*(A)>0$, let $\mathcal{M}_A$ denote the set of all μ_A^*-measurable sets. Prove that $\mathcal{M}=\bigcap\{\mathcal{M}_A : \mu^*(A)>0\}$.

(7) Let μ and ν be positive σ-finite measures on a measurable space $(X,\mathcal{M})$. Suppose that $\nu\ll\mu$ and $\frac{d\nu}{d\mu}>0$ μ-a.e. in X. Prove that $\mu\ll\nu$ and that $\frac{d\mu}{d\nu}=\left(\frac{d\nu}{d\mu}\right)^{-1}$ μ-a.e. and ν-a.e. in X.

(8) Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\in L^1(\mu)$ be given. Define the set function $\nu:\mathcal{M}\to\mathbb{R}$ by $\nu(E):=\int_E f\,\mathrm{d}\mu$ for each $E\in\mathcal{M}$.
a) Verify that ν is a signed measure on X.
b) Provide, with justification, the Jordan decomposition of ν.
c) Identify all null sets for ν.