Real Analysis Comprehensive Examination–Math 921/922

Friday, January 19, 2007, 2:00-6:00p.m., Burnett Hall 203

- Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
- Throughout the exam, the Lebesgue measure is denoted by m and $\mathcal{B}_X$ denotes the Borel σ-algebra on a metric space X.

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(1) a) (10 points) Let $E \subset [0,1]$ be a Lebesgue measurable set with $m(E) = 1$. Prove that E is dense in $[0,1]$.
b) (10 points) Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ and assume that μ is locally finite, i.e., for every $x \in \mathbb{R}$ there exists an open set $E \subset \mathbb{R}$ such that $x \in E$ and $\mu(E) < \infty$. Prove the $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}$.

(2) Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue measurable function and let $g_n(x) = (\sin f(x))^{2n}$, $x \in [0,1]$, $n \in \mathbb{N}$.
a) Carefully explain why g_n is Lebesgue measurable for every $n \in \mathbb{N}$.
b) Show all technical details in evaluating: $\lim_{n\to\infty} \int_{[0,1]} g_n(x)dm$.

(3) Prove that Fatou's Lemma and the Monotone Convergence Theorem are equivalent.

(4) a) Let $T : L^3(\mathbb{R}, m) \longrightarrow \mathbb{C}$ be given by $T(f) := \int_{[0,4]} xf(x)dm(x)$. Prove that T is a bounded linear functional on $L^3(\mathbb{R}, m)$ and find $\|T\|$.
b) Let $(X, \mathcal{M}, \mu)$ be a measure space. Give the definition of $L^\infty(\mu)$, and directly from the definition prove that: If $f, g \in L^\infty(\mu)$ then $f + g \in L^\infty(\mu)$ and $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

(5) Let $f : \mathbb{R} \longrightarrow [0,\infty)$ be Borel measurable and define $\phi : (0,\infty) \longrightarrow [0,\infty]$ by $\phi(y) = m\{x \in \mathbb{R} : \ f(x) > y\}$. Prove that: $\int_{\mathbb{R}} f(x)^2 \, dm(x) = 2\int_{(0,\infty)} y\, \phi(y)\, dm(y)$.

(6) Let X be a compact Hausdorff topological space with the following property:
If $E \subseteq X$ is open then its closure $\overline{E}$ is also open. Let $C(X)$ denotes the space of all continuous $\mathbb{C}$-valued functions on X. Given $f \in C(X)$ and $\epsilon > 0$, prove that there exist $\lambda_1, ..., \lambda_n \in \mathbb{C}$, $E_1, ..., E_n \subseteq X$ such that $\chi_{E_j} \in C(X)$ for every j and $\sup_{x\in X} \left| f(x) - \sum_{j=1}^n \lambda_j \chi_{E_j}(x) \right| < \epsilon$.

(7) a) Let $f \in L^1_{loc}(\mathbb{R}^n)$ and f is continuous at $x_0 \in \mathbb{R}^n$. Prove that $x_0 \in L_f$, where L_f is the Lebesgue set of f.
b) Consider the measure spaces $([0,1], \mathcal{B}_{[0,1]}, m)$ and $([0,1], \mathcal{B}_{[0,1]}, \mu)$, where μ is counting measure on $\mathcal{B}_{[0,1]}$. Prove that $m \ll \mu$, but $dm \neq f d\mu$ for any function f. Does this contradict the Radon-Nikodym Theorem? why?

(8) Let $f, f_n : [0,1] \to \mathbb{R}$ such that $f, f_n \in L^2([0,1], \mathcal{B}_{[0,1]}, m)$, for every $n \in \mathbb{N}$. Prove or disprove the following statements:
a) If $f_n \to 0$ a.e. $[0,1]$ then $\|f_n\|_2 \to 0$.
b) If $f_n \to 0$ in $L^2([0,1], m)$, then $f_n \to 0$ in measure.
c) If $f_n \to f$ weakly in $L^2([0,1], m)$ and $\|f_n\|_2 \to \|f\|_2$, then $\|f_n - f\|_2 \to 0$.
d) If $\|f_n - f\|_2 \to 0$, then $\|f_n - f\|_p \to 0$ for every $p \in [1,2]$.