1) Let \((X, \mathcal{S}, \mu)\) be a measure space, and let \(f \in L^1(X, \mu)\). Define a function \(\nu\) on \(\mathcal{S}\) by \(\nu(A) := \int_A f \, d\mu\). Prove that \(\nu\) is a signed measure on \(X\). Give formulas for \(\nu^+(A), \nu^-(A)\), and \(|\nu|(A)\) (in terms of \(f\)), being sure to completely justify your formulas.

2) Let \((X, \mathcal{A}, \mu)\) be a measure space and suppose \((Y, \mathcal{B})\) is a measurable space. Let \(\varphi : X \to Y\) be a function such that \(\varphi^{-1}(E) \in \mathcal{A}\) whenever \(E \in \mathcal{B}\).
   a) (5 points) For each \(E \in \mathcal{B}\), let \(\nu(E) = \mu(\varphi^{-1}(E))\). Prove that \(\nu\) is a measure on \((Y, \mathcal{B})\).
   b) (5 points) For each measurable function \(f : Y \to \mathbb{R}\), prove that \(f \circ \varphi\) is a measurable function on \(X\).
   c) (10 points) Prove that for all \(f \in L^1(Y, \nu)\), we have
   \[
   \int_Y f \, d\nu = \int_X (f \circ \varphi) \, d\mu.
   \]

3) Suppose \(E \subseteq [0, 1]\) is such that for every interval \(I \subseteq [0, 1]\), \(m^*(I) = m^*(I \cap E) + m^*(I \cap E^c)\). Prove \(E\) is (Lebesgue) measurable.

4) Suppose that \((X, \mu)\) and \((Y, \nu)\) are \(\sigma\)-finite measure spaces.
   a) (4 points) Show that if \(h, k\) belong to \(L^2(Y, \nu)\), then
   \[
   \left| \int_Y h(y)k(y) \, d\nu(y) \right| \leq \|h\|_{L^2(Y)} \|k\|_{L^2(Y)}.
   \]
   b) (12 points) Suppose now that \(K \in L^2(X \times Y, \mu \times \nu)\). If \(g \in L^2(Y, \nu)\) prove that for almost every \(x \in X\),
   \[
   \int_Y |K(x, y)g(y)| \, d\nu(y) < \infty.
   \]
   c) (4 points) Let \(f(x) = \int_Y K(x, y)g(y) \, d\nu(y)\). Prove that \(f \in L^2(X, \mu)\) and that
   \[
   \|f\|_{L^2(X)} \leq \|K\|_{L^2(X \times Y)} \|g\|_{L^2(Y)}.
   \]

5) Suppose \((X, d)\) is a compact metric space, and let \(C(X)\) be the collection of real-valued continuous functions on \(X\). Recall that any compact metric space is separable (don’t prove this) and let \(\{x_n\}_{n=1}^\infty\) be a countable dense subset of \(X\). Put \(f_n(x) = d(x, x_n)\) and let \(\phi\) be the function on \(X\) given by \(\phi(x) = 1\) for every \(x \in X\). Let \(\mathcal{A}\) be the smallest subalgebra of \(C(X)\) containing each \(f_n\) and \(\phi\).
   a) (7 points) If \(h \in C(X)\) and \(\varepsilon > 0\), prove there exists \(g \in \mathcal{A}\) such that for every \(x \in X\), \(|h(x) - g(x)| < \varepsilon\).
   b) (5 points) Describe the form of a typical function in \(\mathcal{A}\).
   c) (8 points) For \(f, g \in C(X)\), let \(\Delta(f, g) := \sup_{x \in X} |f(x) - g(x)|\). Then \((C(X), \Delta)\) is a separable metric space. (The issue is the separability—no points will be awarded for proving \(\Delta\) is a metric, so don’t bother proving \(\Delta\) is a metric.)
6) Let \( F(t) = \int_{[0, \infty)} e^{-xt} \frac{\sin x}{x} \, dm(x) \).
   
   a) (9 points) Prove that \( \lim_{t \to \infty} F(t) = 0 \).
   
   b) (9 points) Prove that for \( t > 0 \), \( F'(t) = -\frac{1}{1 + t^2} \).
   
   c) (2 points) Prove that for \( t > 0 \), \( F(t) = \frac{\pi}{2} - \arctan t \).

7) Let \( f_0 : [a, b] \to \mathbb{R} \) be an increasing function (so \( a \leq x < y \leq b \) implies \( f_0(x) \leq f_0(y) \)). Extend \( f_0 \) to a function \( f : \mathbb{R} \to \mathbb{R} \) by defining

\[
f(x) = \begin{cases} 
  f_0(a) & \text{if } x < a \\
  f_0(x) & \text{if } x \in [a, b] \\
  f_0(b) & \text{if } x > b 
\end{cases}
\]

Let \( g_n(x) := n(f(x + 1/n) - f(x)) \) (where \( n \in \mathbb{N} \)).

   a) (2 points) What can be said about the existence of \( f'(x) \)?
   
   b) (18 points) Use Fatou's Lemma applied to the sequence \( g_n \) to show that \( f' \in L^1[a, b] \) and

\[
\int_a^b f' \, dm \leq f(b) - f(a).
\]

8) Let \( P \) be the set of all irrational numbers, viewed as a metric space with the usual Euclidian metric. For each \( n \in \mathbb{N} \), let \( G_n \subseteq P \) be a (relatively) open subset of \( P \) which is also dense in \( P \). Prove that \( \cap_{n=1}^{\infty} G_n \) is dense in \( P \).