

## Real Analysis Comprehensive Examination

Thursday, January 19, 2006, 2:00–6:00p.m., Oldfather 309

- Work 6 of the 8 problems below.
- Each problem is worth 20 points.
- Write on one side of the paper only.
- Throughout the exam, Lebesgue measure is denoted by  $m$ .

1) Let  $(X, \mathcal{S}, \mu)$  be a measure space, and let  $f \in L^1(X, \mu)$ . Define a function  $\nu$  on  $\mathcal{S}$  by  $\nu(A) := \int_A f d\mu$ . Prove that  $\nu$  is a signed measure on  $X$ . Give formulas for  $\nu^+(A)$ ,  $\nu^-(A)$ , and  $|\nu|(A)$  (in terms of  $f$ ), being sure to completely justify your formulas.

2) Let  $(X, \mathcal{A}, \mu)$  be a measure space and suppose  $(Y, \mathcal{B})$  is a measurable space. Let  $\varphi : X \rightarrow Y$  be a function such that  $\varphi^{-1}(E) \in \mathcal{A}$  whenever  $E \in \mathcal{B}$ .

- a) (5 points) For each  $E \in \mathcal{B}$ , let  $\nu(E) = \mu(\varphi^{-1}(E))$ . Prove that  $\nu$  is a measure on  $(Y, \mathcal{B})$ .
- b) (5 points) For each measurable function  $f : Y \rightarrow \mathbf{R}$ , prove that  $f \circ \varphi$  is a measurable function on  $X$ .
- c) (10 points) Prove that for all  $f \in L^1(Y, \nu)$ , we have

$$\int_Y f d\nu = \int_X (f \circ \varphi) d\mu.$$

3) Suppose  $E \subseteq [0, 1]$  is such that for every interval  $I \subseteq [0, 1]$ ,  $m^*(I) = m^*(I \cap E) + m^*(I \cap E^c)$ . Prove  $E$  is (Lebesgue) measurable.

4) Suppose that  $(X, \mu)$  and  $(Y, \nu)$  are  $\sigma$ -finite measure spaces.

- a) (4 points) Show that if  $h, k$  belong to  $L^2(Y, \nu)$ , then  $\left| \int_Y h(y)k(y) d\nu(y) \right| \leq \|h\|_{L^2(Y)} \|k\|_{L^2(Y)}$ .
- b) (12 points) Suppose now that  $K \in L^2(X \times Y, \mu \times \nu)$ . If  $g \in L^2(Y, \nu)$  prove that for almost every  $x \in X$ ,  $\int_Y |K(x, y)g(y)| d\nu(y) < \infty$ .
- c) (4 points) Let  $f(x) = \int_Y K(x, y)g(y) d\nu(y)$ . Prove that  $f \in L^2(X, \mu)$  and that

$$\|f\|_{L^2(X)} \leq \|K\|_{L^2(X \times Y)} \|g\|_{L^2(Y)}.$$

5) Suppose  $(X, d)$  is a compact metric space, and let  $C(X)$  be the collection of real-valued continuous functions on  $X$ . Recall that any compact metric space is separable (don't prove this) and let  $\{x_n\}_{n=1}^\infty$  be a countable dense subset of  $X$ . Put  $f_n(x) = d(x, x_n)$  and let  $\mathcal{A}$  be the smallest subalgebra of  $C(X)$  containing each  $f_n$  and  $\phi$ .

- a) (7 points) If  $h \in C(X)$  and  $\varepsilon > 0$ , prove there exists  $g \in \mathcal{A}$  such that for every  $x \in X$ ,  $|h(x) - g(x)| < \varepsilon$ .
- b) (5 points) Describe the form of a typical function in  $\mathcal{A}$ .
- c) (8 points) For  $f, g \in C(X)$ , let  $\Delta(f, g) := \sup_{x \in X} |f(x) - g(x)|$ . Then  $(C(X), \Delta)$  is a separable metric space. (The issue is the separability—no points will be awarded for proving  $\Delta$  is a metric, so don't bother proving  $\Delta$  is a metric.)

6) Let  $F(t) = \int_{[0,\infty)} e^{-xt} \frac{\sin x}{x} dm(x)$ .

a) (9 points) Prove that  $\lim_{t \rightarrow \infty} F(t) = 0$ .

b) (9 points) Prove that for  $t > 0$ ,  $F'(t) = -\frac{1}{1+t^2}$ .

c) (2 points) Prove that for  $t > 0$ ,  $F(t) = \frac{\pi}{2} - \arctan t$ .

7) Let  $f_0 : [a, b] \rightarrow \mathbb{R}$  be an *increasing* function (so  $a \leq x < y \leq b$  implies  $f_0(x) \leq f_0(y)$ ). Extend  $f_0$  to a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by defining

$$f(x) = \begin{cases} f_0(a) & \text{if } x < a \\ f_0(x) & \text{if } x \in [a, b] \\ f_0(b) & \text{if } x > b \end{cases}.$$

Let  $g_n(x) := n(f(x + 1/n) - f(x))$  (where  $n \in \mathbb{N}$ ).

a) (2 points) What can be said about the existence of  $f'(x)$ ?

b) (18 points) Use Fatou's Lemma applied to the sequence  $g_n$  to show that  $f' \in L^1[a, b]$  and

$$\int_a^b f' dm \leq f(b) - f(a).$$

8) Let  $P$  be the set of all irrational numbers, viewed as a metric space with the usual Euclidian metric. For each  $n \in \mathbb{N}$ , let  $G_n \subseteq P$  be a (relatively) open subset of  $P$  which is also dense in  $P$ . Prove that  $\bigcap_{n=1}^{\infty} G_n$  is dense in  $P$ .