Your Exam ID Code: ____________________

Real Analysis Comprehensive Examination–Math 921/922
Thursday, January 20, 2005, 2:00-6:00p.m., Avery Hall 112

• Work 6 out of 8 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Throughout the exam, the Lebesgue measure is denoted by m and $(X, \mathcal{M}, \mu)$ denotes a general measure space.

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(1) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n, f : X \to \mathbb{C}$ be $\mathcal{M}$-measurable functions.
a) Give the definition of the following:
(i) $f_n \to f$ in $L^p(\mu)$, for some $p \geq 1$ (ii) $\{f_n\}$ converges to f in measure.
b) Prove that $f_n \to f$ in $L^p(\mu)$ for some $p \geq 1$ implies $\{f_n\}$ converges to f in measure; **but** not conversely.

(2) Let X be a nonempty set.
a) Give the definition of an outer measure μ^* on the set X; and carefully state the meaning of the statement: $A \subset X$ is μ^*-measurable.
b) Let μ^* be an outer measure on X, and let A, B be disjoint μ^*-measurable subsets of X. Prove that for any $E \subseteq X$, $\mu^*(E \cap (A \cup B)) = \mu^*(E \cap A) + \mu^*(E \cap B)$.

(3) a) State the Monotone Convergence Theorem.
b) Let f be a nonnegative function and that $f \in L^1(\mathbb{R}, m)$. Prove that the function $F(x) := \int_{(-\infty,x]} f\, dm$ is continuous on $\mathbb{R}$.

(4) Prove that there exist two Lebesgue measurable sets A and B with the properties: $A \cup B = [0,1]$, $A \cap B = \emptyset$, A is first category, and $m(B) = 0$. (Hint: Consider generalized cantor sets in the interval $[0,1]$.)

(5) Show all technical details in evaluating: $\displaystyle \lim_{n\to\infty} \int_{[0,\infty)} \frac{nx \sin(nx)}{1+x^2}\, e^{-(n+1)x}\, dm(x)$.

(6) Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$. Let $\mu = m$, and ν be the counting measure. Let D be the diagonal in $X \times Y$, (i.e., $D = \{(x,x) : x \in [0,1]\}$) and χ_D denotes the characteristic function on D. Prove that the integrals

$$\int_{[0,1]} \int_{[0,1]} \chi_D\, d\mu d\nu, \quad \int_{[0,1]} \int_{[0,1]} \chi_D\, d\nu d\mu, \text{ and } \int_{[0,1]\times[0,1]} \chi_D\, d(\mu \times \nu)$$

are all unequal, and explain why this result is not a contradiction to Tonelli's Theorem.

(7) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^p(X,\mu) \cap L^\infty(X,\mu)$ for some $1 \leq p < \infty$. Prove that $f \in L^q(X,\mu)$ for every $q > p$ and $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

(8) Let

$$F(x) = \begin{cases} 0; & -\infty < x < 0 \\ 2x+1; & 0 \leq x < 2 \\ 6; & 2 \leq x < \infty \end{cases}$$

a) Show $F \in NBV$, i.e., F is normalized of bounded variation.
b) Let μ_F be the unique Borel measure inherited from F, and $d\mu_F = d\lambda + g dm$ be its Lebesgue decomposition with respect to the Lebesgue measure m.
(i) What is g?
(ii) Find the values of $\lambda(\{0\})$, $\lambda(1,3)$ and $\lambda(a,b)$, where $0, 2 \notin [a,b]$.