Math 901–902 Comprehensive Exam
May 2013

Instructions: Do two problems from each of the three sections, for a total of six problems.
If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

1. Part I: Group and Character Theory

I.1 Let $G$ be a finite group of order $n = |G|$.
   (a) Show that the symmetric group $S_n$ contains a subgroup isomorphic to $G$.
   (b) Show, by providing an example with justification, that the alternating group $A_n$ need not contain a subgroup isomorphic to $G$.
   (c) Show that the alternating group $A_{2n}$ contains a subgroup isomorphic to $G$.
I.2 Find the full character table, with justification, of $D_{10}$, the dihedral group of order 10 (i.e., the group of symmetries of the pentagon).
I.3 Prove the following two statements are equivalent. (You are not allowed to use the Feit-Thompson Theorem.)
   (a) Every finite group of odd order is solvable.
   (b) Every finite simple non-abelian group has even order.

2. Part II: Field and Galois Theory

II.4 Let $K \subseteq F$ be a Galois extension of fields of degree 60. Prove that if $E_1, E_2$ are intermediate fields with $[E_i : K] = 15$, for $i = 1, 2$, then there is a $\sigma \in \text{Gal}(F/E)$ such that $\sigma(E_1) = E_2$.
II.5 Let $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ be field extensions (where $\mathbb{Q}$ denotes the rational numbers and $\mathbb{C}$ denotes the complex numbers). Assume $E$ is an algebraic extension of $\mathbb{Q}$ and that every non-constant polynomial $f \in \mathbb{Q}[x]$ has a root in $E$. Prove that $E$ is the algebraic closure of $\mathbb{Q}$. Hint: Apply the Primitive Element Theorem. (You may use the Primitive Element Theorem without proof.)
II.6 Let $E$ be the splitting field of $x^5 - 2$ over $\mathbb{Q}$. Find, with justification, all intermediate fields $\mathbb{Q} \subseteq L \subseteq E$ such that $[L : \mathbb{Q}] = 5$.
II.7 Recall that a field is perfect if every algebraic extension of it is separable. Prove every algebraic extension of a finite field is perfect.

3. Part III: Ring and Module Theory

III.8 Prove that if $R$ is a semi-simple ring and $a, b \in R$ satisfy $ab = 1$, then $ba = 1$.
III.9 Let $R$ be a commutative ring, let $P$ and $Q$ be finitely generated projective $R$-modules, and let $f : P \to Q$ be an $R$-module homomorphism. Show $f$ is an isomorphism if and only if $f \otimes 1 : P \otimes_R R/m \to Q \otimes_R R/m$ is an isomorphism for every maximal ideal $m$ of $R$.
III.10 Let $R$ be a commutative ring, let $f \in R$, and assume $I$ is an injective $R$-module. Define $\mu_f : I \to I$ to be the $R$-module homomorphism given as multiplication by $f$; i.e., $\mu_f(x) = fx$.
   (a) Prove that if $f$ is a non-zero-divisor, then $\mu_f$ is surjective.
   (b) Prove, by way of example, that if we assume only that $f \neq 0$, then $\mu_f$ need not be surjective.