Math 901–902 Comprehensive Exam

June 2, 2009, 2–6 pm

Instructions: Do two problems from each of the three sections, for a total of six problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Part I: Groups and Character Theory

(I.1) Let \( G \) be a group of order 21.

(a) Determine all possible values of \( n \), where \( n \) is the number of conjugacy classes of \( G \).
   (i.e., for each positive integer \( n \), determine whether or not there is a group of order 21 having exactly \( n \) conjugacy classes.)

(b) Determine the possible decompositions of the group ring \( \mathbb{C}[G] \) as a product of matrix rings. (i.e., find all possible products \( \Pi_{i=1}^r \mathbb{M}_{n_i}(D_i) \) which are isomorphic as rings to \( \mathbb{C}[G] \) for some group \( G \) of order 21, where \( \mathbb{C} \) is the field of complex numbers, each \( D_i \) is a division ring, and \( \mathbb{M}_{n_i}(D_i) \) is the ring of \( n_i \times n_i \) matrices with entries in \( D_i \).)

(I.2) For this problem you may assume that any finite group having a solvable quotient by a solvable normal subgroup is solvable. You may assume Sylow’s Theorems. You may assume that if the order of a finite group \( G \) is the product of at most three (possibly non-distinct) primes, then \( G \) is solvable. You may assume that the commutator subgroup of a group is normal. You may also assume that \( A_n \) is simple for each \( n > 4 \). Prove everything else that you use or claim in your arguments.

(a) If \( p \) is prime and \( i \geq 0 \) is an integer, prove that a group \( G \) of order \( p^i \) is solvable.

(b) Find the least positive integer \( n \) such that there is a non-solvable group of order \( n \). Justify your answer; in particular, justify that all groups of order less than \( n \) are solvable.

(I.3) Find all integers \( 0 < n < 20 \) such that there exists a non-nilpotent group of order \( n \). Justify your answer. (You may assume whatever general facts you know about nilpotent groups, but explicitly state any fact you use. If you claim a particular group is or is not nilpotent, either give a proof or cite a general fact that justifies your claim.)

Part II: Fields and Galois Theory

(II.4) Let \( F/k \) be an extension of fields such that \( |k| = 3^6 \) and \( |F| = 3^{60} \). How many elements \( \alpha \in F \) are there with \( k(\alpha) = F \)? Justify your answer.

(II.5) Let \( F \) be a finite (but not necessarily Galois) extension of a field \( K \). Given a subgroup \( G < \text{Aut}_K(F) \), let \( F^G \) denote the intermediate field \( \{ f \in F : g(f) = f \text{ for all } g \in G \} \).
(a) Let $G < \text{Aut}_K(F)$ be a subgroup. Prove Artin’s theorem that $F/F^G$ is a finite Galois extension with Galois group $G$.

(b) Let $S$ be the set of subgroups of $\text{Aut}_K(F)$ and let $I$ be the set of intermediate fields of the extension. Define a map $\phi : S \rightarrow I$ for any $G \in S$ by setting $\phi(G) = F^G$. Show that $\phi$ is always injective and give an example to show that $\phi$ need not be surjective.

(II.6) Determine the group $\text{Aut}(\mathbb{R})$ of field automorphisms of the reals. [Hint: Prove for any automorphism $\sigma$ that $\sigma(x) < \sigma(y)$ for any reals $x < y$.]

Part III: Rings and Modules

(III.7) Let $G$ be a finite group and let $\mathbb{C}$ denote the field of complex numbers. Show that $R = \mathbb{C}[G]$ has a nonzero nilpotent element if and only if $G$ is a nonabelian group.

(III.8) Let $A$ be a commutative ring with $1 \neq 0$. Let $M$ and $N$ be $A$-modules. Show that $M \oplus N$ is flat if and only if $M$ and $N$ are flat.

(III.9) Compute the number of elements in the $\mathbb{Z}$-module $A \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(M \oplus N, P \oplus Q)$, where $A = S^{-1}\mathbb{Z}$, $S = \{1, 5, 5^2, \ldots\}$, $M = \mathbb{Z}/9\mathbb{Z}$, $N = \mathbb{Z}/5\mathbb{Z}$, $P = \mathbb{Z}/3\mathbb{Z}$ and $Q = \mathbb{Z}/25\mathbb{Z}$. Justify your answer.