Math 901–902 Comprehensive Exam

May 31, 2006, 2–6pm

Solve two problems from each of the three parts, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. **Justify all your answers**.

Part A.

(1) Suppose G is a finite simple group with $|G| \geq 3$ and $H \leq G$ is a proper subgroup of index n. Prove $|G|$ divides $\frac{n!}{2}$.

(2) Let G be a finite group, let p be a prime dividing $|G|$, and let P be a Sylow p-subgroup of G.

(a) Prove P is the unique Sylow p-subgroup of $N_G(P)$. (Here, for a subgroup $H \leq G$, we write $N_G(H)$ for the normalizer of H in G.)

(b) Prove $N_G(P) = N_G(N_G(P))$.

(3) This problem concerns groups of order 105.

(a) Prove any group of order 105 contains a normal subgroup of order 35.

(b) Prove there are just two groups, up to isomorphism, of order 105.

Part B.

(4) Suppose $k \subset L$ is a finite, Galois extension with Galois group isomorphic to A_5.

(a) Prove there is no intermediate field $k \subset E \subset L$ with $[E : k]$ equal to 2, 3, or 4.

(b) Find, with proof, the number of intermediate fields $k \subset E \subset L$ with $[E : k] = 12$.

(5) Let $f(x) = x^4 + 3 \in \mathbb{Q}[x]$ and let L be the splitting field of $f(x)$. Find, with justification, the Galois group $Gal(L/\mathbb{Q})$.

(6) Let $f(x) \in \mathbb{Q}[x]$ be a degree five polynomial, let L be the splitting field of $f(x)$, and let $G = Gal(L/\mathbb{Q})$.

(a) Prove $f(x)$ is irreducible if and only if five divides $|G|$.

(b) Assume that $f(x)$ is irreducible. Prove $|G|$ can be equal to 5. (*Hint*: One approach is to consider a subextension of a cyclotomic extension.)

Part C.

(7) Let R be a left semi-simple ring.

(a) Prove every quotient ring of R (i.e., every ring of the form R/I where I is a two-sided ideal) is also a left semi-simple ring.

(b) Give, with justification, an example showing that a subring of R need not be a left semi-simple ring.

(8) Let ρ be a complex, linear representation of a finite group G, and let χ be the associated character.

(a) Prove $\frac{1}{|G|}\sum_g \chi(g)$ is the number of times the trivial one-dimensional representation appears in the direct sum decomposition of ρ into irreducible representations.

(b) Prove that if $\frac{1}{|G|}\sum_g \|\chi(g)\|^2 = 3$, then ρ is a direct sum of three *distinct* irreducible representations.

(9) Let $N = \langle x \rangle$ be a cyclic group of order 3, let $H = \langle y \rangle$ be a cyclic group of order 4, and let

$$G = N \rtimes_\alpha H,$$

where $\alpha : H \to \mathrm{Aut}(N)$ is given by $\alpha(y)(x) = x^{-1}$. (In terms of generators and relations, $G = \langle x, y \mid x^3 = e, y^4 = e, yxy^{-1} = x^{-1} \rangle$.) You may use, without proof, that the conjugacy classes of G are represented by the elements e, y^2, x, xy^2, y, y^3 and have sizes $1, 1, 2, 2, 3, 3$ respectively.

(a) Prove that there are, up to isomorphism, precisely two irreducible 2-dimensional (complex linear) representations of G.

(b) Describe one such irreducible 2-dimensional representation of G. (*Hint*: One way to do this is to consider the group $G/\langle y^2 \rangle$.)

(c) Give, with justification, the character table of G.