Solve two problems from each of the three parts, for a total of six problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. **Justify all your answers.**

**Part A.**

1. Suppose $G$ is a finite simple group with $|G| \geq 3$ and $H \leq G$ is a proper subgroup of index $n$. Prove $|G|$ divides $\frac{n!}{2}$.

2. Let $G$ be a finite group, let $p$ be a prime dividing $|G|$, and let $P$ be a Sylow $p$-subgroup of $G$.

   (a) Prove $P$ is the unique Sylow $p$-subgroup of $N_G(P)$. (Here, for a subgroup $H \leq G$, we write $N_G(H)$ for the normalizer of $H$ in $G$.)

   (b) Prove $N_G(P) = N_G(N_G(P))$.

3. This problem concerns groups of order 105.

   (a) Prove any group of order 105 contains a normal subgroup of order 35.

   (b) Prove there are just two groups, up to isomorphism, of order 105.

**Part B.**

4. Suppose $k \subset L$ is a finite, Galois extension with Galois group isomorphic to $A_5$.

   (a) Prove there is no intermediate field $k \subset E \subset L$ with $[E : k]$ equal to 2, 3, or 4.

   (b) Find, with proof, the number of intermediate fields $k \subset E \subset L$ with $[E : k] = 12$.

5. Let $f(x) = x^4 + 3 \in \mathbb{Q}[x]$ and let $L$ be the splitting field of $f(x)$. Find, with justification, the Galois group $Gal(L/\mathbb{Q})$. 
(6) Let \( f(x) \in \mathbb{Q}[x] \) be a degree five polynomial, let \( L \) be the splitting field of \( f(x) \), and let \( G = \text{Gal}(L/\mathbb{Q}) \).

(a) Prove \( f(x) \) is irreducible if and only if five divides \( |G| \).

(b) Assume that \( f(x) \) is irreducible. Prove \( |G| \) can be equal to 5.  
(Hint: One approach is to consider a subextension of a cyclotomic extension.)

Part C.

(7) Let \( R \) be a left semi-simple ring.

(a) Prove every quotient ring of \( R \) (i.e., every ring of the form \( R/I \) where \( I \) is a two-sided ideal) is also a left semi-simple ring.

(b) Give, with justification, an example showing that a subring of \( R \) need not be a left semi-simple ring.

(8) Let \( \rho \) be a complex, linear representation of a finite group \( G \), and let \( \chi \) be the associated character.

(a) Prove \( \frac{1}{|G|} \sum_g \chi(g) \) is the number of times the trivial one-dimensional representation appears in the direct sum decomposition of \( \rho \) into irreducible representations.

(b) Prove that if \( \frac{1}{|G|} \sum_g \|\chi(g)\|^2 = 3 \), then \( \rho \) is a direct sum of three distinct irreducible representations.

(9) Let \( N = \langle x \rangle \) be a cyclic group of order 3, let \( H = \langle y \rangle \) be a cyclic group of order 4, and let \( G = N \rtimes \alpha H \), where \( \alpha : H \to \text{Aut}(N) \) is given by \( \alpha(y)(x) = x^{-1} \).  
(In terms of generators and relations, \( G = \langle x, y \mid x^3 = e, y^4 = e, yxy^{-1} = x^{-1} \rangle \).) You may use, without proof, that the conjugacy classes of \( G \) are represented by the elements \( e, y^2, x, xy^2, y, y^3 \) and have sizes 1, 1, 2, 2, 3, 3 respectively.

(a) Prove that there are, up to isomorphism, precisely two irreducible 2-dimensional (complex linear) representations of \( G \).

(b) Describe one such irreducible 2-dimensional representation of \( G \).  
(Hint: One way to do this is to consider the group \( G/ \langle y^2 \rangle \).

(c) Give, with justification, the character table of \( G \).