Math 901–902 Comprehensive Exam

June 1, 2004, 1–5pm

Do two problems from each of the three sections, for a total of *six* problems.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

A. Group Theory

1. Let G be a group whose center has index n. Prove that G contains at most n^2 distinct commutators.

2. Classify up to isomorphism all groups of order 70. Be as complete in your analysis as possible.

3. Let C_2 denote the group of 2 elements, let D_m denote the dihedral group of order $2m$, and let n be an odd number.

   (a) Prove that the groups D_{2n} and $D_n \times C_2$ are isomorphic.

   (b) Prove that the groups D_{4n} and $D_{2n} \times C_2$ are not isomorphic.

B. Field and Galois Theory

4. Let F be a finite field with q elements. Prove that an irreducible polynomial $f(x) \in F[x]$ divides $x^{q^n} - x \in F[x]$ if and only if $\deg(f)$ divides n.

5. Let E denote the field $\mathbb{Q}(\zeta + \zeta^{-1})$, where $\mathbb{Q}$ is the field of rational numbers and $\zeta \in \mathbb{C}$ is a primitive nth root of 1. Determine the Galois group $G(E/\mathbb{Q})$.

   [Hint: Embed E into the cyclotomic field $\mathbb{Q}(\zeta)$.]

6. Let $\mathbb{Q}$ denote the field of rational numbers.

   (a) Let $E/\mathbb{Q}$ and $F/\mathbb{Q}$ be finite Galois extensions of $\mathbb{Q}$ such that $E \cap F = \mathbb{Q}$. Prove that $\mathrm{Gal}(EF/\mathbb{Q}) \cong \mathrm{Gal}(E/\mathbb{Q}) \times \mathrm{Gal}(F/\mathbb{Q})$.

   (b) Let G be a cyclic group of odd order. Prove there exists a finite Galois extension E of $\mathbb{Q}$ such that $\mathrm{Gal}(E/\mathbb{Q}) \cong G$.

   [Hint: Consider Sylow subgroups of the Galois groups of cyclotomic extensions.]

C. Rings and Modules

Throughout this section all rings are assumed to be commutative with identity.

7. Let R be an integrally closed domain which is not a field. Prove that the quotient field K of R is not algebraically closed.

   [Hint: In fact, show that K is not closed under taking square roots.]

8. Let R be a Noetherian ring and M a finitely generated R-module. Prove that $\{p \in \operatorname{Spec} R \mid M_p \text{ is free } \}$ is an open subset of $\operatorname{Spec} R$. [Hint: If M_p is a free R_p-module prove there exists an element $a \notin p$ such that M_a is a free R_a-module.]

9. Let R be a Noetherian ring and M a finitely generated R-module.

(a) Prove there exists a finite filtration of R-submodules of M

$$0 = M_0 \subset M_1 \subset M_2 \cdots \subset M_n = M$$

such that for $i = 1, \ldots, n$, there exists a prime ideal p_i of R such that $M_i/M_{i-1} \cong R/p_i$.

(b) Given a filtration for M as in part (a), prove that for each $p \in \operatorname{Ass}_R M$, there exists an index i such that $M_i/M_{i-1} \cong R/p$. [Hint: Recall that if $0 \to A \to B \to C$ is exact, then $\operatorname{Ass}_R A \subseteq \operatorname{Ass}_R B \subseteq \operatorname{Ass}_R A \cup \operatorname{Ass}_{\mathrm{R}} C$.]