

Math 901–902 Comprehensive Exam
June 6, 1–5pm

Do two of the three given problems from each of the three sections, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

I Group Theory

- I.1 Let G be a group and G' the commutator subgroup of G (also called the derived subgroup of G).
- (a) Let H be a subgroup of G . Prove that $H \supseteq G'$ if and only if $H \triangleleft G$ and G/H is abelian.
 - (b) Let N be a normal subgroup of G such that $N \cap G' = \{1\}$. Prove that $N \subseteq Z(G)$. (Here, $Z(G)$ denotes the center of G .)
- I.2 Let p be a prime with $p \equiv 1 \pmod{4}$. Prove there are at least three isomorphism classes of nonabelian groups of order $4p$. (Your solution should include a justification of why these classes are distinct.)
- I.3 Let G be a finite group, p a prime dividing $|G|$, and P a Sylow p -subgroup of G . Let Q be any p -subgroup of G . Prove that $P \cap Q = N_G(P) \cap Q$. (Here, $N_G(P)$ denotes the normalizer of P in G .)

II Field and Galois Theory

- II.1 Let E/K be a finite Galois field extension. Suppose F is an intermediate field of E/K having the property that for all other intermediate fields L such that $[F : K]$ divides $[L : K]$, we have $F \subseteq L$. Prove that F/K is normal.
- II.2 Let K be a field and $f(x)$ a separable irreducible polynomial in $K[x]$. Let E be a splitting field for $f(x)$ over K and $\alpha, \beta \in E$ distinct roots of $f(x)$. Suppose $K(\alpha) = K(\beta)$. Let $G = \text{Aut}(E/K)$ and $H = \text{Aut}(E/K(\alpha))$.
- (a) Prove that $H \neq N_G(H)$.
 - (b) Use (a) to deduce that if $\deg f$ is prime then G is cyclic.
- II.3 Let \mathbb{F}_q be a field with q elements and $f(x) \in \mathbb{F}_q[x]$ an irreducible polynomial of degree m . Prove that for any positive integer n , $f(x)$ divides $x^{q^n} - x$ if and only if m divides n .

III Rings and Modules

Note: Throughout this section R is assumed to be a commutative ring with identity.

III.1 Let R be a domain and

$$0 \rightarrow K \rightarrow P_\ell \rightarrow P_{\ell-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0$$

an exact sequence of R -modules. Suppose that for $i = 0, \dots, \ell$, P_i is a finitely generated projective R -module of rank r_i . Prove that K is a finitely generated projective R -module and

$$\text{rank } K = \sum_{i=0}^{\ell} (-1)^{\ell-i} r_i.$$

(Recall that if R is a domain and P is projective, then $\text{rank } P = \dim_F S^{-1}P$, where S is the set of non-zero elements of R and $F \cong S^{-1}R$ is the field of fractions of R .)

- III.2 Let R be a ring and $f : M \rightarrow N$ an R -module homomorphism. Suppose there exists elements $r_1, \dots, r_k \in R$ such that $(r_1, \dots, r_k) = R$ and $\frac{f}{r_i} : M_{r_i} \rightarrow N_{r_i}$ is an isomorphism for all i . Prove that f is an isomorphism. (For $r \in R$, M_r denotes the localization of M at the multiplicatively closed set $\{r^n \mid n \geq 0\}$.)
- III.3 Let R be a Noetherian domain and Q the field of fractions of R . Let $u \in Q$. Prove that u is integral over R if and only if there exists $r \in R, r \neq 0$ such that $ru^n \in R$ for all $n \geq 1$. (Hint: For the reverse implication, consider the R -submodule of Q generated by $\frac{1}{r}$.)