Do two problems from each of the three sections, for a total of six problems.

I Group Theory

I.1 Classify all groups of order 2001 up to isomorphism. (By the way, 2001 = 3 \cdot 23 \cdot 29.)

I.2 Suppose a finite group \( G \) acts transitively on a finite set \( S \) with \( |S| \geq 2 \). Prove there is a \( g \in G \) such that \( g \cdot s \neq s \) for all \( s \in S \).

I.3 Let \( H \) and \( K \) be subgroups of a finite group \( G \). Suppose \( [G: H] = p \), where \( p \) is prime, and \( p \) is strictly smaller than every prime divisor of \( |K| \). Prove \( K \leq H \).

II Field Theory and Galois Theory

II.1 Suppose \( K \subset E_1 \) and \( K \subset E_2 \) are finite Galois field extensions. (Assume \( E_1 \) and \( E_2 \) are each contained in some fixed algebraic closure \( K \) of \( K \).) Prove \( [E_1E_2 : K] \mid [E_1 : K] \cdot [E_2 : K] \).

II.2 Let \( F \) be the splitting field of \( f(x) = x^4 + 4x^2 + 2 \) over \( \mathbb{Q} \). Find the lattice of subfields of \( F \). (Hint: If \( \pm \alpha, \pm \beta \) are the roots of \( f(x) \), consider \( \frac{\alpha^2 - \beta^2}{\alpha \beta} \).)

II.3 Suppose \( K \subset F \) is a finite Galois field extension and \( p^n \mid [F : K] \), where \( p \) is prime. Prove that there exists an intermediate field extension \( K \subset E \subset F \) such that \( [F : E] = p^n \).

III Rings and Modules

III.1 Let \( R \) be a commutative ring (with 1) and suppose \( S \subset R \) a multiplicatively closed subset. Suppose the natural ring map \( R \to S^{-1}R \) defines an integral ring extension. (In particular, assume the map \( R \to S^{-1}R \) is injective.) Prove that \( R \to S^{-1}R \) is an isomorphism.

III.2 Let \( R \) be a commutative ring (with 1) and assume every \( R \)-module is injective.

(a) Prove that if \( R \) is an integral domain, then \( R \) must be a field.

(b) Give an example showing that the integral domain hypothesis in (a) is necessary.

III.3 Let \( A \) and \( B \) be commutative rings (with 1) and let \( A \to B \) be a ring map. Suppose \( P \) is a projective \( A \)-module and prove \( P \otimes_A B \) is a projective \( B \)-module.