Math 901–902 Comprehensive Exam

17th January 2006, 2–6pm

Solve two problems from each of the three parts, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Justify all your answers.

Part A.

(1) Let G be a finite group, with n conjugacy classes. For each $x \in G$, let $C(x)$ denote the centralizer of x in G.

(a) Prove the identity: $\sum_{x\in G} |C(x)| = n|G|$.

(b) What is the probability that two randomly chosen elements in G commute?

(2) Prove that a group of order $450 = 2 \cdot 3^2 \cdot 5^2$ cannot be simple.

(3) Let G be a group of order 75. Prove the following statements.

(a) If G is non-abelian, then it contains a subgroup isomorphic to $\mathbb{Z}/5 \times \mathbb{Z}/5$.

(b) There are at least 3 non-isomorphic groups, possibly abelian, of order 75.

Part B.

(4) Let α be a root of the polynomial $f(x) = x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$.

(a) Prove that $-3\alpha^2 - 2\alpha + 9 = (2\alpha^2 + \alpha - 3)^2$ in the field $\mathbb{Q}[\alpha]$.

(b) Prove that $\mathbb{Q}[\alpha]$ is the splitting field of $f(x)$ over $\mathbb{Q}$.
Hint: One approach is to factor $f(x)$ over $\mathbb{Q}[\alpha]$.

(5) Let $K \subseteq F$ be an extension of fields and $\sigma\colon F \to F$ a homomorphism of fields which fixes K.

(a) When F is an algebraic extension of K, prove that σ is an isomorphism.

(b) Give an example of a σ which is not an isomorphism.

(6) Compute the Galois group over $\mathbb{Q}$ of the equation $x^4 - 19$.

Part C.

(7) Let F be a field of characteristic p (possibly, $p = 0$), and let n be a positive integer. Prove that the ring $F[x]/(x^n - 1)$ is semi-simple if and only if p does not divide n.

(8) Let R be a simple artinian ring, and I a minimal left ideal in R.

(a) Prove that there exist elements $a_1, \ldots, a_n$ in I and $r_1, \ldots, r_n$ in R such that $1 = a_1 r_1 + \cdots + a_n r_n$.

(b) Prove that there exists an integer n such that $R \cong I^{\oplus n}$ as left R-modules. Here $I^{\oplus n}$ denotes the direct sum of n copies of I.

(9) This question concerns the representations of A_4, the alternating group on four elements, over $\mathbb{C}$.

(a) Write down a $\mathbb{C}$-vectorspace basis for the center of the group algebra $\mathbb{C}[A_4]$.

(b) How many irreducible representations does A_4 have, and what are their degrees?

(c) Write down (with justifications!) the character table of A_4.