Math 901–902. Comprehensive Exam
January 18, 2005; 2 p.m.–6 p.m.

Do two problems from each one of Sections G, F, and R, for a total of *six* problems. When working on a question in a problem with multiple questions you can use the assertions of earlier questions.

G. Groups. In this section G is a group and $\mathrm{Aut}(G)$ its group of automorphisms.

G1. Prove that if $|G| = 616$, then G is solvable. [Note: $616 = 2^3 \cdot 7 \cdot 11$]

G2. Assume $|G| = 4n + 2$ for some integer $n \geq 0$. Let S_m denote the symmetric group on m elements, and let $\iota\colon G \to S_{4n+2}$ be the homomorphism of groups given by the action of G on itself by left multiplication.

(a) Prove that if $g \in G$ has order 2, then $\iota(g)$ is an odd permutation.

(b) Prove that G contains a normal subgroup of order $2n + 1$.

G3. Prove that if $\mathrm{Aut}(G) = 1$, then $|G| \leq 2$.

F. Fields and Galois Theory. In this section L denotes a field.

F1. Let L be the field of rational fractions $\mathbb{F}_p(x, y)$, where p is a prime number, and F be the subfield $\mathbb{F}_p(x^p, y^p) \subseteq L$. For each integer $n \geq 1$ consider the element $z_n = x + x^{p^n} y \in L$ and the subfield $E_n = F(z_n) \subseteq L$. Prove the following assertions.

(a) $(L : F) = p^2$ and $(E_n : F) = p$.

(b) $E_n \neq E_m$ if $n \neq m$.

F2. Let L be algebraically closed of characteristic 0, let $\varphi\colon L \to L$ be an automorphism, and set $F = \{x \in L \mid \varphi(x) = x\}$. Prove that every finite field extension K of F is cyclic. [Hint: Use a normal closure E of K over F.]

F3. Find, with justification, a primitive element over $\mathbb{Q}$ for $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

R. Rings and Representations. In this section R is ring with $1 \neq 0$, G is a finite group, and representations of G are over the field of complex numbers $\mathbb{C}$.

R1. State and prove the Hilbert Basis Theorem.

R2. Let F a free R-module of rank n. Suppose $x_1, \ldots, x_n$ generate F. Prove that $\{x_1, \ldots, x_n\}$ is a basis for F.

R3. Let R be a commutative ring. A prime ideal p of R is said to be *minimal* if there is no prime ideal of R properly contained in p. Prove the following assertions.

(a) R has at least one minimal prime ideal.

(b) If R is Noetherian, then it has only finitely many minimal prime ideals.

R4. Let R be an artinian semisimple ring. Prove the following assertions.

(a) If $x, y \in R$ satisfy $xy = 1$, then $yx = 1$.

(b) If $I = xR$ is a two-sided ideal of R, then $I = Rx$.

R5. For $G = S_3$ write down the irreducible representations, prove that your list is complete, and give the character table.

R6. Let g, h be elements of G. Prove the following assertions.

(a) g and h are conjugate if and only if $\chi(g) = \chi(h)$ for every character χ of G.

(b) g is conjugate to g^{-1} if and only if $\chi(g) = \chi(g^{-1})$ for every character χ of G.