Math 901–902 Comprehensive Exam

January 20, 2004, 2–6pm

Do two problems from each of the three sections, for a total of *six* problems. If you work on more than two problems from any section, be sure to say which you want counted.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

A. Group Theory

1. Let G be a finite group of odd order, and let H be a subgroup of index 5. Prove that H is normal in G. (Hint: Use that S_5 has no element of order 15.)
2. Let G be a simple group of order less than 60. Prove that G is cyclic.
3. Let G be a group of order $5^2 \cdot 29^2$.
   (a) Suppose the Sylow 29-subgroup of G is cyclic. Prove that G is abelian.
   (b) Prove that there exists a nonabelian group of order $5^2 \cdot 29^2$. (You do not need to find a presentation for such a group.)
4. Prove that there is no simple group of order 90. (Hint: Focus on the Sylow 3-subgroups.)

B. Field and Galois Theory

5. Prove that $\mathbb{C}$, the field of complex numbers, is algebraically closed. (Give an algebraic proof, using Sylow theorems and Galois theory. The only analytic tools you should need are the following facts: (a) Every real polynomial of odd degree has a real root; and (b) every complex number has a square root.)
6. Let $K/\mathbb{Q}$ be a finite Galois extension whose Galois group is non-abelian and simple. Suppose $\alpha^p \in \mathbb{Q}$ where $\alpha \in K$ and p is a prime integer. Prove that $\alpha \in \mathbb{Q}$.
7. Let F be a subfield of $\mathbb{C}$ maximal with respect to not containing $\sqrt{2}$, and let K/F be a finite extension with $K \subseteq \mathbb{C}$. Prove that K/F is Galois and that the Galois group is cyclic of prime power order.
8. Let F be a field of prime characteristic p, and let $f(X) = X^p - X - a \in F[X]$. Let α be a root of f (in an algebraic closure of F). Assuming that $\alpha \notin F$, prove that $F(\alpha)/F$ is Galois of degree p.

C. Rings and Modules

Throughout this section all rings are assumed to be commutative with identity.

9. Let R be a ring, m a maximal ideal of R, and M an R-module. Suppose $m^n M = 0$ for some positive integer n. Prove that M is Noetherian if and only if M is Artinian.
10. Let $R \subseteq T \subseteq Q(R)$ be domains where R is Noetherian, T is a finitely generated R-algebra, and $Q(R)$ is the quotient field of R. Prove that T is integral over R if and only if there exists a nonzero element $r \in R$ such that $rT \subseteq R$.
11. Let R be a Dedekind domain and M a finitely generated R-module. Prove that M is projective if and only if M is torsion-free.
12. Let R be a Noetherian ring, P a prime ideal of R and M a non-zero finitely generated R-module.
    (a) Prove that $P \in \mathrm{Ass}_R(M)$ if and only if $PR_P \in \mathrm{Ass}_{R_P}(M_P)$.
    (b) If P is minimal over the annihilator $(0 :_R M)$ of M, prove that $P \in \mathrm{Ass}_R(M)$.