

Math 901-902 Comprehensive Exam
January 18, 2001

Do two problems from each of the three sections, for a total of six problems.

SECTION I: GROUPS

1. Recall that a group G is *nilpotent* if the central series $C^{(i)}(G) = (e)$ for some i , where $C^{(0)}(G)$ is the center of G and $C^{(i)}(G)$ is the preimage in G of the center of $G/C^{(i-1)}(G)$.
 - (a) Prove that a finite p -group is nilpotent.
 - (b) Prove that a finite direct product of nilpotent groups is nilpotent.
2. Let G be a group of order $5 \cdot 7 \cdot 19^2$.
 - (a) Prove that if the sylow 19-subgroup is cyclic then G is cyclic.
 - (b) Prove there exists a non-abelian group of order $5 \cdot 7 \cdot 19^2$.
3. Suppose that G contains a proper subgroup of finite index.
 - (a) Show that G contains a proper normal subgroup of finite index.
 - (b) For the group $G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Q} \right\}$, find all proper normal subgroups and describe a proper subgroup that is not normal.

SECTION II: RINGS AND MODULES

4. Consider the following commutative diagram of R -modules and R -linear maps. Assume the rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \end{array}$$

- (a) Prove there exists a unique R -module homomorphism $f: A \rightarrow L$ making the diagram commute.
 - (b) Suppose g is surjective and h is injective. Prove that f is surjective.
5. Suppose that R is a ring with identity. Show that a unitary R -module J is injective if and only if for every left ideal L of R , every R -module homomorphism $L \rightarrow J$ may be extended to an R -module homomorphism $R \rightarrow J$.

9. Let R be a commutative ring which is finitely generated as an algebra over a field k . Let M be an R -module of finite length. Prove that M is a finite dimensional k -vector space.

10. Let $R \subset S$ be an integral ring extension where R is Noetherian and S is a finitely generated R -algebra. Let I be an ideal of R and $J = IS \cap R$ (which is also an ideal of R). Prove that there exists an integer k such that for all $n \geq k$, $I^n \subseteq J^n \subseteq I^{n-k}$.

SECTION III: FIELDS AND GALOIS THEORY

3. Suppose that K is an infinite field and L is an extension field of K . Prove that a finite separable field extension has a primitive element.

4. Let ω be a primitive 15th root of unity.

(a) Find the minimal polynomial of ω over \mathbf{Q} .

(b) Find the Galois group of $\mathbf{Q}(\omega)/\mathbf{Q}$.

(c) Find field generators for all the subfields of $\mathbf{Q}(\omega)$ of degree 4 over \mathbf{Q} .

5. Let E/K be a separable field extension of degree p , where p is a prime. Suppose $f(x) \in K[x]$ is an irreducible polynomial which has more than one root in E . Prove that $f(x)$ splits in E .