Ph.D. Comprehensive Exam
Algebra: 901-902. January 23, 1997.

Do a total of six problems: three from Sections I and II combined, and three from sections III and IV combined. Do no more than two problems from any one section. In sections III and IV all rings are assumed to be commutative with identity.

Section I: Groups

1. Classify all groups of order 63 up to isomorphism.

2. Show that the multiplication table for a group determines a presentation for the group. That is, let F denote the free group on the set $\{x_g \mid g \in G\}$, $\phi\colon F \to G$ the group homomorphism defined by $\phi(x_g) = g$ and $H = \ker \phi$. Prove that H is the smallest normal subgroup containing all the words of the form $x_a x_b x_c^{-1}$ where $ab = c$ as elements of G.

3. Compute $\mathrm{Aut}(S_3)$.

Section II: Fields

4. Let K be a field and f and g be irreducible separable polynomials in $K[x]$. Let F_f and F_g be the splitting fields for f and g, respectively. Consider the statement "f is irreducible over F_g if and only if g is irreducible over F_f."

(a) Show that the statement is false. (Hint: look at $x^3 - 2$ and $x^2 + 3$.)
(b) In the case that $\mathrm{Aut}(F_f/K)$ and $\mathrm{Aut}(F_g/K)$ are cyclic, prove that the statement is true.

5. Let F/K be a finite extension of fields. Let P be the fixed field of $\mathrm{Aut}(F/K)$ and let S/K be the maximal separable extension of K contained in F. Assume that P/K is purely inseparable.

(a) Prove that S/K is Galois.
(b) Prove that $\mathrm{Aut}(S/K)$ is isomorphic to $\mathrm{Aut}(F/K)$.

6. Let F/K be a finite dimensional Galois extension and E an intermediate field. Let $G = \mathrm{Aut}(F/K)$ and $H = \mathrm{Aut}(F/E)$. Prove that $\mathrm{Aut}(E/K) \cong N_G(H)/H$.

Section III: General rings and modules

7. Let R be a reduced ring (i.e., no non-zero nilpotents) and suppose we have an injective map of R-modules $f\colon R^m \to R^n$. Prove that $m \le n$.

8. Let R be a ring and $X = \mathrm{Spec}(R)$ endowed with the Zariski topology.

(a) Prove that X is compact. (Recall that a topological space τ is *compact* if every open covering of τ has a finite subcovering.)
(b) Give an example of a ring R where X is not T_1. (A topological space τ is T_1 if given any two distinct points $x, y \in \tau$ there exists an open set O which contains x but not y.)

9. Let R be a ring and M and N two R-modules. Consider the statement "if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$."

(a) Show that the statement is false in general.
(b) Prove that if R has a unique maximal ideal and M and N are finitely generated as R-modules, then the statement is true.

Section IV: Noetherian rings and modules

10. Let $R \subset S$ be an integral extension of commutative rings where R is Noetherian and S is a finitely generated R-algebra. Prove that for each prime ideal p of R there exist only finitely many prime ideals q of S such that $q \cap R = p$.

11. Let R be a Noetherian ring. Prove that every irreducible ideal of R is primary.

12. Let R be a Noetherian ring and M a finitely generated R-module. Prove that $\{p \in \mathrm{Spec}\, R \mid M_p \text{ is a free } R\text{-module}\}$ is an open subset of $\mathrm{Spec}\, R$.