Work of NU Professor Emeritus William G. Leavitt is foundation for exciting new area of research

The Winter 2010 NU Math News contained an article describing the outstanding contributions and accomplishments of NU's Commutative Algebra group. But don't get the wrong impression ... NU NON-Commutative Algebra is having quite a worldwide impact too!

Here's some mathematical background. If you think back to your Intro to Modern Algebra Math 310 course, most likely all of the rings you saw as basic examples have what's known as the "Invariant Basis Number" property. This means that if you have a finitely generated free module M over R (the moral equivalent of a finite dimensional vector space over R), then any two bases for M must have the same number of elements. Thinking back even further, you probably saw a proof of this fact in your introduction to linear algebra course, in the case that R is the field $\mathbb{R}$ of real numbers. More formally, the Invariant Basis Number property for a ring R means that for every pair of positive integers m and n, if the free left R-modules ${}_RR^m$ and ${}_RR^n$ are isomorphic, then $m = n$. It is not too hard to show that not only does this IBN property hold for fields, it actually holds for all commutative rings, all (finite-sized) matrix rings over fields, and most if not all of the rings you saw in Math 310.

In 1962, NU Professor William G. Leavitt wrote an article for the prestigious journal *Transactions of the American Mathematical Society*, in which he showed that there is a plethora of rings which do NOT have IBN. Specifically, Bill proved this:

Theorem. For any pair of positive integers m, n with $m < n$ and any field K there exists a K-algebra $L_K(m, n)$ with the following properties: for each positive integer $i < m$, and each positive integer $j \neq i$, the free left R-modules ${}_RR^i$ and ${}_RR^j$ are not isomorphic; on the other hand, for each positive integer $i \geq m$, the free left modules ${}_RR^i$ and ${}_RR^j$ are isomorphic precisely when $i \equiv j \pmod{n - m}$.

Less formally, what Professor Leavitt's Theorem shows is that, in the context of isomorphisms between free modules, *anything that can happen, does happen.*

Over the next three-plus decades this interesting work received somewhat modest attention in the mathematics community. But in the past seven years, the attention level surrounding these algebras has risen from modest to *intense*! Here's why. In 2004 a handful of algebraists came up with a construction which associates with any finite directed graph E and field K a K-algebra, denoted $L_K(E)$, whose definition is based on the configuration of vertices and edges in E. As it turns out, if you start with the directed graph R_n having one vertex and $n \geq 2$ loops based at that vertex, then $L_K(R_n)$ is precisely the algebra $L_K(1, n)$ that Bill Leavitt constructed and investigated back in the sixties!

These newfangled algebras are called *Leavitt path algebras* in honor of Bill Leavitt's foundational work in the area. (Yes, the L in $L_K(E)$ stands for 'Leavitt'.) Many mathematicians throughout the world have been focused on these $L_K(E)$ algebras for the past seven years. Interest in them comes not only from the noncommutative algebra community, but from the operator algebra community as well, because there is an intimate connection between Leavitt path algebras with coefficients in the complex numbers $K = \mathbb{C}$, and analytic structures called *graph C^*-algebras*.

During April 2011 a pair of talks were delivered at NU in which an update on some of the current happenings in the field of Leavitt path algebras were discussed. How nice it was that Bill Leavitt, at age 95 and still going strong, could attend!

NU's Professor Emeritus William G. Leavitt's impact on mathematics research will endure well into the future.