A NONLINEARIZABLE CUBIC-LINEAR MAPPING

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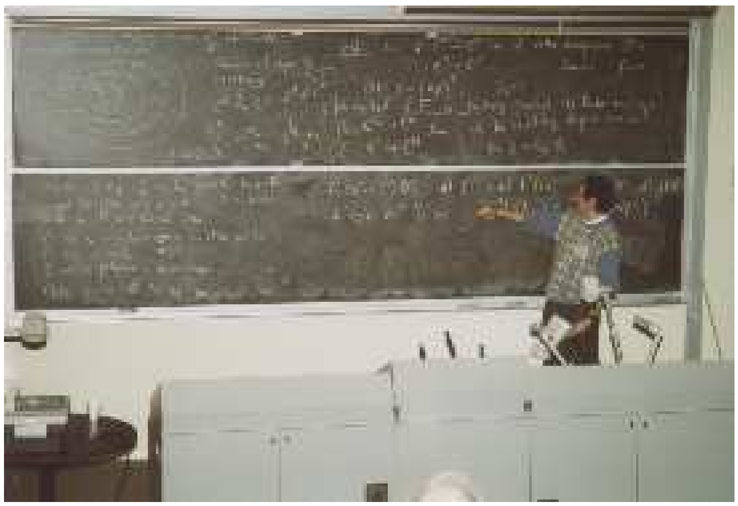

Arno van den Essen in his paper [3] produces, among other things, the following cubic-homogeneous polynomial automorphism of $\mathbb{C}^5$:

$$f\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} x_2 x_5^2 \\ x_1^2 x_5 - x_4 x_5^2 \\ x_2^2 x_5 \\ 2x_1 x_2 x_5 - x_3 x_5^2 \\ 0 \end{pmatrix} \tag{0.1}$$

that has the following property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $k_\lambda\colon \mathbb{C}^5 \to \mathbb{C}^5$ that linearizes λf, in the sense that $\lambda f(k_\lambda(x)) = k_\lambda(\lambda x)$ for all $x \in \mathbb{C}^5$. It is therefore a counterexample to a conjecture that originated in [1].

The same paper in theorem 2.3 states that there exists a cubic-linear polynomial automorphism F of $\mathbb{C}^{17}$ with the same property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $K_\lambda\colon \mathbb{C}^{17} \to \mathbb{C}^{17}$ such that $\lambda F(K_\lambda(X)) = K_\lambda(\lambda X)$ for all $X \in \mathbb{C}^{17}$. This F is then a counterexample to a more special conjecture that

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was advanced in [5]. However the F is not actually exhibited, but only a hint at its construction is given, following the general method described in [4].

Now we have carried out all the calculations and here are the results. The map F is defined as $F(X) := X - (AX)^{*3}$, where the exponent means the componentwise cubic power, and the matrix A is given as

$$A = \frac{1}{12}\begin{pmatrix}
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & -12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & -12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & -12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & 12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & 12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It can be checked that $A^2 = 0$. Consider also the following two matrices B:

$$B = \frac{1}{12}\begin{pmatrix}
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{pmatrix}$$

and C (shown here transposed to save space):

$$C^T = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

It can be verified that the product BC is the 5×5 identity matrix, that A and B have the same kernel, and that if we define $F(X) := X - (AX)^{*3}$ for $X \in \mathbb{C}^{17}$, we have that $f(x) = BF(Cx)$ for all $x \in \mathbb{C}^5$. This means that the mappings f and F are "paired" in the sense of [4].

It follows that the mapping F is a cubic-linear automorphism of $\mathbb{C}^{17}$ such that λF is not linearizable for any $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$.

We have implemented the procedure for finding A, B, C from f, that was described in [4], as the following routine written in the *Mathematica* programming language, version 3.0, by Wolfram Research Inc.

```
makePairing[cubicHomogeneous_Function]:=
  Module[{dimension,var,toCubeCombination,recombined,
      monomialList,combinationList,temp,comb,
      d0,b0,b,d,c,m,cm,dAnd0,a,answer},
dimension=Length[Last[cubicHomogeneous]];
var=Table[x[i],{i,dimension}];
SetAttributes[toCubeCombination,Listable];
toCubeCombination[a_+b_]:=toCubeCombination[a]+
  toCubeCombination[b];
toCubeCombination[0]:=0;
toCubeCombination[(num_.)*(a_)*(b_)*(c_)]:=
```

```
(num/24)*(a+b+c)^3+(num/24)*(a-b-c)^3-(num/24)*(a+b-c)^3-
     (num/24)*(a-b+c)^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!NumberQ[c];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3+(num/6)*(a-b)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3-(num/6)*(b-a)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)^3]:=
      num*a^3/;NumberQ[num]&&!NumberQ[a];
recombined=toCubeCombination[Evaluate[
      Expand[cubicHomogeneous@@var-var]]];
monomialList=Map[If[Head[#]===Plus,List@@#,{#}]&,
      recombined]//Flatten;
combinationList=
      Union[Select[monomialList,
            MatchQ[#1, (num_.)*(lin_)^3] & ]/.
            (num_.)*(lin_)^3:>lin/;NumberQ[num]];
temp=recombined/.
        Table[combinationList[[i]]^3->comb[i],
        {i,Length[combinationList]}];
    d0=Table[Coefficient[combinationList[[i]],x[j]],

{i,Length[combinationList]},{j,dimension}];
b0=-Table[
          Coefficient[temp[[i]],comb[j]],{i,dimension},{j,
            Length[combinationList]}];
For[i=0,Union[Flatten[Minors[b0,dimension]]]=={0},i++,
b0=Transpose[
          Join[Transpose[b0],
             {IdentityMatrix[dimension][[dimension - i]]}]];
d0=Join[d0,{Table[0,{dimension}]}]];
b=b0;
d=d0;
c= Module[{c,mat},
  mat=Array[c,{Length[b[[1]]],dimension}];
   mat/.
   Solve[b.mat==IdentityMatrix[dimension],
       Flatten[mat]][[1]]/.c[i_,j_]->0];
m= Transpose[NullSpace[b]];
m=m*LCM @@ Denominator[Union[Flatten[m]]];
cm=Transpose[Join[Transpose[c], Transpose[m]]];
dAnd0 =Transpose[
```

```
        Join[Transpose[d],
          Transpose[Table[0,{i,Length[d]},
          {j,Length[First[m]]}]]]]];
a=dAnd0.Inverse[cm];
answer=(b.c==IdentityMatrix[dimension])&&(
          Union[Expand[cubicHomogeneous@@var-var+
              b.(a.c.var)^3]]=={0})&&(
          Union[Flatten[a.Transpose[NullSpace[b]]]]=={0})&&

(Union[Flatten[b.Transpose[NullSpace[a]]]]=={0});
If[answer,{a,b,c},Print["Something is wrong"]]];
```

The function `makePairing` takes a cubic-homogeneous function f from $\mathbb{C}^n$ to itself, with $n \geq 2$, and returns a triple of matrices A, B, C such that $\ker A = \ker B$, $BC = I_n$ and $f(x) = x - B(ACx)^{*3}$. If f has constant Jacobian determinant so has the mapping $X \to X - (AX)^{*3}$, and f is an automorphism if and only if F is (in the respective dimensions).

The routine has been tested only with the following three cubic-homogeneous examples, the first of which is the (0.1) above and the others are taken from [3] and [4]:

```
f = Function[{x1, x2, x3, x4, x5},
  {x1 + x2 * x5^2, x2 + x1^2 * x5 - x4 * x5^2, x3 + x2^2  *x5,
   x4 + 2 x1 * x2 * x5 - x3 * x5^2, x5}];
g = Function[{x1, x2, x3, x4},
  {x1 + (x3*x1 + x4 * x2) * x4,
   x2 - (x3*x1 + x4*x2) * x3, x3 + x4^3, x4}];
h = Function[{y1, y2, y3, y4, y5},
  {y1, y2, y3, y4, y5} + 3 * {0, 0,
    y1^2 * y2  +y1 * y2^2  +2 * y1 * y2 * y4 - 2 * y1^2 * y5,
    - y1^2 * y2 - y1 * y2^2  -2y1 * y2 * y3 - 2 * y1^2 * y5,
    - y2^2 * y3 - y2^2 * y4}];
```

We cannot guarantee that the algorithm will not run into bugs in other cases, but there is a built-in check that should warn if the results are not reliable. The command to give is simply `makePairing[f]`.

An electronic copy of a *Mathematica* notebook containing the routine should be available on the Web together with these proceedings.

References

1. B. Deng, G. Meisters, G. Zampieri, *Conjugation for polynomial mappings*, Z, Angew. Math. Phys. 46 (1995).
2. A. van den Essen (ed.), *Automorphisms of Affine Spaces*, Proceedings of the conference "Invertible Polynomial Maps", held in Curaçao, July 4-8, 1994, under auspices of the Caribbean Mathematical Foundation, Kluwer Academic Publisher, Dordrecht/Boston/London, 1995.
3. A. van den Essen, *A counterexample to Meister's Cubic-Linear Linearization Conjecture*, Preprint, University of Nijmegen, to appear in Indagationes Mathematicae.

4. G. Gorni, G. Zampieri, *On cubic-linear polynomial mappings*, Research Report, University of Udine (August 1996), to appear in Indagationes Mathematicae.
5. G. Meisters, The Cubic-Linear Linearization Conjecture. This paper is available since November 1995 on the World-Wide-Web at the address `http://w.w.w.math.unl.edu/~gmeister`.