SOME REMARKS ON CONTINUOUS AND DISCRETE MARKUS-YAMABE PROBLEMS

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Dedicated to Gary H. Meisters

Abstract. The aim of this paper paper is twofold. By one side it summarizes some of the results of the three authors (some of them made also in collaboration with A. van den Essen, E. Hubbers and J. Llibre, see the references [Cima & others]) related with injectivity and Markus-Yamabe problems. On the other hand we point out some open problems related with the above papers, discussing finally which is the relation between injectivity and global asymptotic stability.

1 Introduction

Before starting with a brief survey about the actual situation of Markus-Yamabe (MY) problems we would like to stress how the work of Gary Meisters and Czeslaw Olech has strongly influenced our interest into them. Our first contact with the

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problems was motivated by a visit of J. Sotomayor to Barcelona around 1984. In that visit he explained us the MY Conjecture and gave us a copy of [20]. From that moment we started to be interested in the MY problems. After several years appeared the famous paper [17]. From that paper and again motivated by a visit of J. Sotomayor to Barcelona, appeared the paper [13]. R. Fessler and C. Gutierrez explained us that they were also motivated by [17] and [13] to continue the study of MY Conjecture. In fact they separately proved it for the plane, see [12, 14]. At this point M. Sabatini organized a meeting in Trento. Later A. van den Essen organized another meeting in Curaçao. It was in both places were we enter in contact with most people interested in MY problems and also was the starting point of the paper [4]. In this last paper the authors presented a polynomial counterexample to the MY Conjecture for dimension greater than two.

Here we describe which are the MY problems.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. We say that F satisfies:

(i) The *Markus-Yamabe assumption*, (MYA), if for any $x \in \mathbb{R}^n$, the jacobian of F at x has all its eigenvalues with negative real part.

(ii) The *discrete Markus-Yamabe assumption*, (DMYA), if for any $x \in \mathbb{R}^n$, $DF(x)$ has all its eigenvalues with modulus less than one.

Markus-Yamabe problems.

(i) **MYC(n)** *(Markus-Yamabe Conjecture, [19]). Let F be a $\mathcal{C}^\infty$ vector field defined on $\mathbb{R}^n$ satisfying the MYA. If $F(p) = 0$, then the critical point p is a global attractor of $\dot{x} = F(x)$.*

(ii) **DMYQ(n)** *(Discrete Markus-Yamabe Question, [7, 22]). Let F be a $\mathcal{C}^\infty$ map from $\mathbb{R}^n$ into itself such that $F(0) = 0$ and satisfying the DMYA. Is it true that the fixed point 0 is a global attractor for the discrete dynamical system generated by F?*

As it has already been told, the MYC is true for $n = 2$ and false for $n \geq 3$ (even for polynomial F), see for instance [4] and the references therein, or even better the Web Address [16].

The present situation of the DMYQ is the following: The answer is yes for $n = 1$ and no for $n \geq 2$. In fact for $n = 2$ the answer is yes for polynomial F, while the example which has not 0 as a global attractor is given by a rational map. For $n \geq 3$ it is possible to give a polynomial F without any global attractor. See [4, 7] and the references therein.

In the first part of this paper (Sections 2, 3 and 4) we present a survey of several results about the MYC and the DMYQ, specially the results which are related with the notion of "quasi-homogeneity" of polynomials. At the same time we present some related open problems which we think that are interesting by themselves. Although they will be stated more precisely in the paper we list them here:

(1) Is the MYC(n) generically true for polynomial vector fields ? (Problem 2).

(2) Are there quasi-homogeneous linear maps $N : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with $DN(x)$ nilpotent and with all its n components linearly independent over $\mathbb{C}$? (Problem 1).

(3) Which is the answer to the DMYQ for entire maps $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$? (Problem 3).

As we will see in Section 3.2 the answer to Question 2 is relevant for knowing if there are polynomial MY vector fields with periodic orbits or invariant tori. It is well known that there are smooth non polynomial counterexamples with a periodic orbit, see [2, 3].

This part of the paper is organized as follows. Section 2 deals with quasi-homogeneous objects and their properties. The proof of these properties can be found in [8]. Section 3 is devoted to the continuous case (MYC) while in Section 4 we treat with the discrete case (DMYQ).

The last part of the paper is included in Section 5. There we study the relation between injectivity and the existence of critical points which are global attractors. As far as we know the results of this section are new.

2 On quasi-homogeneous vector fields of degree one

This section is a summary of some results of [8]. See also [1, Chap. 1]. It is said that $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is a *quasi-homogeneous* function with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* d if

$$f(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_n}x_n) = \lambda^d f(x_1, x_2, \ldots, x_n) \quad , \quad \text{for all } \lambda > 0.$$

The weights can be taken as non zero real numbers.

It is said that $F = (F_1, F_2, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a *quasi- homogeneous* vector field (resp. map) with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* d if each F_i is a quasi-homogeneous function with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* $\alpha_i + d - 1$. Quasi-homogeneous vector fields (resp. maps) of degree one are called *linear quasi-homogeneous vector fields (resp. maps).* From now on we deal with them.

Given the weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ we define the "semi straight line"which passes through a point $x \in \mathbb{R}^n$, as

$$L_x = \{(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_n}x_n) : \lambda \in \mathbb{R}^+\}.$$

We note that if $\alpha_i\alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, then 0 belongs to the limit set of L_x. If the weights have different sign then, in general, it is not so. The above fact makes a big difference between the two different situations.

Finally, consider $\dot{x} = F(x)$ and $x \in \mathbb{R}^n$. The solution which passes through x is called of *exponential type* if

$$x(t) = (x_1 e^{m_1 t}, x_2 e^{m_2 t}, \ldots, x_n e^{m_n t})$$

for some $m_1, m_2, \ldots, m_n \in \mathbb{R}$.

Proposition 1 *([8]) Let F be a linear quasi-homogeneous vector field with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and consider the differential system $\dot{x} = F(x)$. The following hold:*

(i) *F is invariant by the change of variables $\bar{x}_i = \lambda^{\alpha_i}x_i$.*

(ii) *If $\alpha_i\alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, then the knowledge of the orbits near the origin determines the global phase portrait of F. Furthermore, if 0 is locally asymptotically stable, then 0 is a global attractor.*

(iii) *Let L_x be the semi straight line which passes through x. Then L_x is invariant by the flow of $\dot{x} = F(x)$ if and only if the solution which passes through x is of the form $x_k(t) = x_k e^{m_k t}$ where $m_k = c\alpha_k$, $k = 1, 2, \ldots, n$, $c \in \mathbb{R}$. Furthermore these solutions of exponential type can be found by solving solve the nonlinear system of equations*

$$F_k(x) = c\alpha_k x_k, \qquad k = 1, 2, \ldots, n$$

where c is a real number.

Observe that a Corollary of (ii) of the above proposition is that if we assume that F is linear quasi-homogeneous with $\alpha_i \alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, and it satisfies the MYA then 0 is a global attractor.

For the case of discrete dynamical systems similar results can be developed. Let F be a linear quasi-homogeneous map and consider the discrete dynamical system

$$x^{(m)} = F(x^{(m-1)}) \quad , \quad x^{(0)} \in \mathbb{R}^n \quad , \quad m \in \mathbb{N}.$$

An orbit of this system is the set $\{x^{(m)} : m \in \mathbb{N}\}$. We say that the orbit which begins at $x^{(0)}$ is of *exponential type* if there exist some constants $a_1, a_2, \ldots, a_n$ such that

$$x^{(m)} = (x_1^{(0)} a_1^m, x_2^{(0)} a_2^m, \ldots, x_n^{(0)} a_n^m).$$

If for $i = 1, 2, \ldots, n$, $|a_i| < 1$ then $\lim_{m\to\infty} x^{(m)} = 0$, while if $|a_i| > 1$ and $x_i^0 \neq 0$ for some $i = 1, 2, \ldots, n$ then $\lim_{m\to\infty} ||x^{(m)}|| = \infty$. A "Semi straight line" L_x is now invariant if $F(L_x) \subset L_x$.

For this type of dynamical systems we have a result similar to Proposition 1. (iii):

Proposition 2 *([8]). Let F be a linear quasi-homogeneous vector field with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and let L_x be the semi straight line which passes through x. Then L_x is invariant by the discrete dynamical system generated by F if and only if the orbit which begins at x is of exponential type. Furthermore to find the invariant straight lines it suffices to solve the system of equations*

$$F_k(x) = \lambda^{\alpha_k} x_k, \qquad k = 1, 2, \ldots, n,$$

where λ is a real positive number.

3 Continuous Case

3.1 Examples of polynomial Markus-Yamabe vector fields having unbounded orbits

The first polynomial counterexample to the MYC which appeared in the literature is the following (see [4]):

$$\begin{cases} \dot{x} = & -x + z(x + yz)^2 \\ \dot{y} = & -y - (x + yz)^2 \\ \dot{z} = & -z \end{cases}$$

This is a linear quasi-homogeneous MY vector field with weights 1,2,-1. To find invariant straight lines we solve the system:

$$\begin{cases} -x + z(x+yz)^2 & = cx \\ -y - (x+yz)^2 & = 2cy \\ -z & = -cz \end{cases}$$

which gives $c = 1$ and the set of solutions $\{(x, \frac{-x^2}{27}, \frac{18}{x}) \, : \, x \in \mathbb{R} \setminus \{0\}\}$. We note that the above set can be described as $L_{\mathbf{x}} \cup L_{\mathbf{x}'}$ where $\mathbf{x} = (18, -12, 1)$ and $\mathbf{x}' = (-18, -12, -1)$. Hence, we obtain the solutions of exponential type $x(t) = \pm 18e^t$, $y(t) = -12e^{2t}$, $z(t) = \pm e^{-t}$. Clearly $||(x(t), y(t), z(t))|| \to \infty$ as $t \to \infty$ and 0 is not a global attractor.

Observe that the main idea of the above example consists into taking linear quasi-homogeneous vector fields of the form $\lambda x + H(x)$ where $H(x)$ is a nilpotent map and $\lambda < 0$. In [8] the authors used this idea to obtain a family of vector fields which contains the above one. In [10] the author is able to find counterexamples of the same form in $\mathbb{R}^n, n \geq 5$ but with $H(x)$ homogeneous of degrees 2 and 3 and linear quasi-homogeneous. These counterexamples are also relevant from the point view of the relation between the MY Conjecture and the Jacobian Conjecture, see [18, 23]. Both families of counterexamples are described in the sequel.

Proposition 3 *([8]). The family of vector fields*

$$\dot{x} = (\lambda x_1 - b x_3^m (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_2 + a x_3^l (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_3, \dots, \lambda x_n)$$

satisfy the following properties:

(1) *They are linear quasi-homogeneous with weights*

$$(\alpha_1, \alpha_2, \dots, \alpha_n) = (m + kl, l + km, 1 - k, \dots, 1 - k),$$

for all $a, b, \lambda \in \mathbb{R}, k, l, m \in \mathbb{N}$.

(2) *They satisfy the MYA, for all $\lambda \in \mathbb{R}$ with $\lambda < 0$.*

(3) *They have unbounded orbits for all $\lambda \in \mathbb{R}$ with $\lambda < 0$, $k \geq 2$ an even number, $l, k, l - m \in \mathbb{N}$ different from zero and for all $a, b \in \mathbb{R}$.*

Inside this family, the vector field which has less degree is obtained by considering $k = 2, m = 0$ and $l = 1$, what gives degree five.

Proposition 4 *([10]). The families of vector fields defined for $x \in \mathbb{R}^n, n \geq 5$,*

$$F_1(x) = -x + Q(x) \qquad \text{and} \qquad F_2(x) = -x + x_5 Q(x),$$

where

$$Q(x) = (x_2 x_5, x_1^2 - x_4 x_5, x_2^2, 2x_1 x_2 - x_3 x_5, 0, \dots, 0),$$

define vector fields which satisfy the MYA and which have orbits that tend to infinity when the time goes to infinity.

3.2 Markus-Yamabe linear quasi-homogeneous vector fields and the existence of periodic orbits

In this section we try to find a polynomial MY vector field with a periodic orbit. To this end we discuss how can be usefull a generalization to the complex of the method used in [8] to obtain MY vector fields with unbounded orbits.

Take $x = (x_1, x_2 \dots x_n) \in \mathbb{C}^n = \mathbb{R}^{2n}$ and a linear quasi-homogeneous vector field with real weights $\alpha_1, \alpha_2 \dots \alpha_n$ and of the form

$$\lambda x + N(x),$$

with $N(x) = (N_1(x), N_2(x), \dots, N_n(x))$ and $DN(x)$ a nilpotent matrix. When $\lambda = -1 + \beta i$, $0 \neq \beta \in \mathbb{R}$ the vector field is a MY vector field. If the system of equations

$$(\alpha_1 x_1, \alpha_2 x_2 \dots, \alpha_n x_n) i = \lambda x + N(x),$$

had some solution $0 \neq x \in \mathbb{C}^n$ then the differential equation

$$\dot{x} = \lambda x + N(x),$$

would be a MY vector fields with particular solutions of the form

$$x(t) = (x_1 e^{i\alpha_1 t}, x_2 e^{i\alpha_2 t}, \dots, x_n e^{i\alpha_n t}).$$

Observe that the above solutions would give either periodic orbits or invariant tori in $\mathbb{R}^{2n}$ for the differential equation considered.

Also note that if $N_1(x), N_2(x), \dots, N_n(x)$ are linearly dependent over $\mathbb{C}$ then, through a linear change of variables, the differential equation can be written as follows:

$$\dot{z} = \lambda z + (M_1(z), M_2(z), \dots, M_{n-1}(z), 0),$$

where $z = (z_1, z_2 \dots, z_n) \in \mathbb{C}^n$ and $M = (M_1, M_2, \dots, M_{n-1}, 0)$ with DM also nilpotent. Then a periodic solution or an invariant torus of the differential equation must be included in the hyperplane $z_n = 0$. Hence we can restrict our attention to a new differential equation in $\mathbb{C}^{n-1}$,

$$\dot{w} = \lambda w + (M_1(w, 0), M_2(w, 0), \dots, M_{n-1}(w, 0)),$$

where $w = (z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ and the map $M^*(w) := (M_1(w, 0), M_2(w, 0), \dots, M_{n-1}(w, 0))$ has again its Jacobian matrix $DM^*(x)$ nilpotent.

The conclusion is that an example in $\mathbb{C}^n$ of the form $\dot{x} = \lambda x + N(x)$ with $N = (N_1, N_2, \dots, N_n)$ such that $DN(x)$ is a nilpotent map and $N_1, N_2, \dots, N_n$ linearly dependent over $\mathbb{C}$ gives an example in dimension $n - 1$.

So to obtain examples in "minimal" dimension we need maps with nilpotent Jacobian matrix which components are linearly independent. It is easy to see that for $n = 2$ there are no such kind of examples. As far as we know, the only examples in the literature, for $n \geq 3$, are constructed by van den Essen in [9]. Unfortunately these examples are not linear quasi-homogeneous. We formulate the following question:

Problem 1 Are there linear quasi-homogeneous maps $N = (N_1, N_2, \dots, N_n) : \mathbb{C}^n \to \mathbb{C}^n$ with nilpotent Jacobian matrix $DN(x)$ and such that $N_1, N_2, \dots, N_n$ are linearly independent over $\mathbb{C}$ for some $n \geq 3$?

3.3 Genericity of the Markus-Yamabe Conjecture

In the set of all polynomial maps of fixed degree we define the following topology. We denote by $\mathcal{X}_m$ the set of all polynomial maps $F = (P^1, P^2, \dots, P^n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\deg(P^i) \leq m$. By identifying $\mathcal{X}_m$ with $\mathbb{R}^M$, where M is the number of all coefficients of $P^1, P^2, \dots, P^n$, we endow $\mathcal{X}_m$ with the so called *coefficient topology.*

Inside $\mathcal{X}_m$ we consider the set of all the maps which satisfy the MYA and we denote by $\text{Int}\{MYA\}$ the interior of this set. The following is an open problem.

Problem 2 If $F \in \text{Int}\{MYA\}$ and $F(0) = 0$, is it true that 0 is globally asymptotically stable?

We think that the answer to this question is very relevant in the setting of the problem. In fact, if we consider the interior the set $\{F \in \mathcal{X}_m : F \text{ satisfies the RJA }\}$ where RJA means the Real Jacobian assumption (i. e., $\det DF(x) \neq 0$, for all $x \in \mathbb{R}^n$), then we know that all the maps in this set are injective (see [5]). Hence, in spite of the fact that the Real Jacobian Conjecture is false (see [21]) it is true for a big subset of maps which satisfy the hypothesis. What about the MY Conjecture?

The first step is to give a characterization of $\text{Int}\{MYA\}$. If $F = (P^1, P^2, \dots, P^n) \in \mathcal{X}_m$, we denote by $F_m = (P^1_m, P^2_m, \dots, P^n_m)$, where P^i_m is the homogeneous part of P^i of degree m. The answer is the following.

Proposition 5 *([5]) The following statements hold.*

(i) *If F satisfies the MYA then $DF_m(x)$ has all the eigenvalues with non-positive real part at each $x \in \mathbb{R}^n$.*

(ii) *Let $F \in \mathcal{X}_m$. Then $F \in \text{Int}\{MYA\}$ if and only if $DF_m(x)$ has all the eigenvalues with negative real part at each $x \neq 0$.*

(iii) *The set $\text{Int}\{MYA\}$ is non empty if and only if m is odd.*

Assuming that F satisfies the MYA, the existence of a critical point globally asymptotically stable implies that $\dot{x} = F(x)$ has a unique critical point and that there are not orbits of $\dot{x} = F(x)$ which scape at infinity.

From Proposition 5 we can see that if $F \in \text{Int}\{MYA\}$ then $F + c \in \text{Int}\{MYA\}$ for all $c \in \mathbb{R}$ and hence the uniqueness of the critical point is equivalent to the injectivity of F. In fact, each $F \in \text{Int}\{MYA\}$ is injective (see Theorem 1).

Concerning the stability of the orbits of $\dot{x} = F(x)$ with F satisfying the MYA, let a be an infinite critical point of $p(F)$ (the Poincaré compactification of F) and let λ_a be the eigenvalue associated to the eigenvector not contained in the tangent space at infinity. It is easy to prove that $\lambda_a < 0$ implies the existence of some unbounded orbit which tends to infinity in the direction determined by a (see [5]).

We summarize the results in the following theorem.

Theorem 1 *([5]) Let $F \in \text{Int}\{MYA\}$ and consider $\dot{x} = F(x)$. Then:*

(i) *$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is bijective.*

(ii) *For each a infinite critical point of F, $\lambda_a > 0$ where λ_a is the eigenvalue associated to the eigenvector not contained in the tangent space at infinity.*

4 The discrete Case

4.1 Examples and counterexamples in dimension two

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map satisfying the DMYA. If F is a polynomial map, then we get the following.

Theorem 2 *([7]) Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a polynomial map satisfying the DMYA. Then it exists a unique fixed point of F which is a global attractor for the discrete dynamical system generated by F.*

The proof of Theorem 2 is based in the fact that each polynomial map satisfying the hypothesis of the theorem, through an affine transformation, is a triangular map. For this kind of maps, the result is true in any dimension (see Theorem A in [7]).

The next question is to know if the result is true for a more general class of maps. The following proposition gives a negative answer for the class of the diffeomorphisms. From the dynamical point of view the class of diffeomorphisms is interesting because we can consider the positive time (the future: $x_1 = F(x_0)$) and the negative time (the past: $x_{-1} = F^{-1}(x_0)$).

Proposition 6 *([7]) Let $G_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*

$$G_a(x,y) = \left(-ax - \frac{ky^3}{1+x^2+y^2}\, ,\, -ay - \frac{kx^3}{1+x^2+y^2}\right)$$

where $k \in \left(1, \frac{2}{\sqrt{3}}\right)$. Then for a small enough, the map G_a is a global diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ which satisfies the following properties:

(i) *For all $x \in \mathbb{R}^n$ and for all λ eigenvalue of $(DG_a)(x)$, $|\lambda| < 1$.*

(ii) *$G_a(0) = 0$ and there exists $q \in \mathbb{R}^n$, $q \neq 0$ which satisfies $G_a^4(q) = q$.*

Another question related with this problem, which has the attention of some mathematicians is the following.

Problem 3 Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be an entire map with all the eigenvalues of $DF(x)$ with modulus less than one at each $x \in \mathbb{C}^2$. Is it true that there exists an unique fixed point of F which is a global attractor for the discrete dynamical system generated by F?

4.2 Examples of polynomial Markus-Yamabe maps having unbounded orbits

The first polynomial map satisfying the DMYA and having unbounded orbits was given by A. van den Essen and E. Hubbers (see [11]) and it is the following:

$$F(x,y,z,w) = \left(\frac{1}{2}x + w(xz+yw)^2, \frac{1}{2}y - (xz+yw)^2, \frac{1}{2}z + w^2, \frac{1}{2}w\right).$$

Inspired in this example we developed the theory of section 2 and it was possible to construct the counterexamples in dimension three.

This is a linear quasi-homogeneous map with weights -5, -4, 2, 1. To find invariant straight lines we solve the system

$$\begin{cases} \frac{1}{2}x + w(xz+yw)^2 & = \lambda^{-5}x \\ \frac{1}{2}y - (xz+yw)^2 & = \lambda^{-4}y \\ \frac{1}{2}z + w^2 & = \lambda^2 z \\ \frac{1}{2}w & = \lambda w \end{cases}$$

which gives $\lambda = \frac{1}{2}$, $x = \frac{(31)^2 63}{(32)^3} w^{-5}$, $y = \frac{31(63)^2}{8(32)^2} w^{-4}$ and $z = -4w^2$. The set of solutions is $L_{\mathbf{x}} \cup L_{\mathbf{x}'}$ where $\mathbf{x} = \left(\frac{(31)^2 63}{(32)^3}, \frac{31(63)^2}{8(32)^2}, -4, 1\right) = (x_0, y_0, z_0, w_0)$ and $\mathbf{x}' = (-x_0, y_0, z_0, -w_0)$. And one unbounded orbit is $\left((32)^m x_0, (16)^m y_0, (\frac{1}{4})^m z_0, (\frac{1}{2})^m w_0\right)$.

In fact, we have the analogous of Proposition 3 for the discrete case.

Proposition 7 *([8]) The family of maps*

$$F(x) = (\lambda x_1 - b x_3^m (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_2 + a x_3^l (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_3, \ldots, \lambda x_n)$$

satisfy the following properties:

(1) *For all $a, b, \lambda \in \mathbb{R}, k, l, m \in \mathbb{N}$ they are linear quasi-homogeneous with weights*

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (m + kl, l + km, 1 - k, \ldots, 1 - k).$$

(2) *For all $\lambda \in \mathbb{R}$ with $\lambda < 0$ they satisfy the DMYA.*

(3) *For all $\lambda \in \mathbb{R}$ with $|\lambda| < 1$, $k \geq 2$ an even number, $l, k, l - m \in \mathbb{N}$ different from zero and for all $a, b \in \mathbb{R}$ the discrete dynamical system generated by F has unbounded orbits.*

Of course a similar result to Proposition 4 holds for iteration of maps.

4.3 Genericity and the relation with the Jacobian Conjecture

The first result about polynomial maps satisfying the DMYA is the following.

Lemma 1 *([7]) Let F be a polynomial map from $\mathbb{R}^n$ to itself such that it satisfies the DMYA. Then the characteristic polynomial of $(DF)_x$ is independent of x.*

Let $F \in \mathcal{X}_m$ be a polynomial map of degree m and assume that F satisfies the DMYA. The above lemma, in particular implies that $\det(DF_m)_x \equiv 0$ for all $x \in \mathbb{R}^n$. It is clear that if we perturb slightly F inside $\mathcal{X}_m$ we obtain some maps which dont satisfy the above property. So, we can assert that $\text{Int}\{DMYA\} = \emptyset$. Hence, in this setting, it has no sense to speak about genericity.

On the other hand, assuming that F satisfies the DMYA, the existence of a fixed point of F being a global attractor for the dynamical system generated by F clearly implies that this fixed point is unique. Hence, we can formulate a problem weaker than the DMYQ as follows:

Conjecture 1 (Fixed Point Conjecture, FPC) *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a polynomial map satisfying the DMYA. Then F has a unique fixed point.*

Next theorem shows that this problem is equivalent to the celebrated Jacobian Conjecture, which can be established as follows.

Conjecture 2 (Jacobian Conjecture, JC) *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map with $detDF(x) \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ at each $x \in \mathbb{C}^n$. Then F is invertible.*

Theorem 3 *([7])*

$$JC \quad \text{is equivalent to} \quad FPC$$

The conclusion is that the DMYA has no global implications from the dynamical point of view but perhaps has implications on the injectivity of certain maps.

To end this section we want to state a result which in some sense is a reciprocal to JC for $\mathcal{C}^\infty$ maps.

Theorem 4 *([6, Thm. D]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ injective map. Then $detDF(x)$ does not change sign.*

5 Global attractors and injectivity

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a $\mathcal{C}^\infty$ map and consider the differential system $\dot{x} = F(x)$. Assume that $F(0) = 0$ and that 0 is a local attractor. In this section we study the problem of which additional conditions imply that 0 is a global attractor. Clearly, if F satisfies the MYA then we know that it is so.

We begin with two simple examples. Consider the following integrable systems:

$$\left\{ \begin{array}{l} \dot{x} = -x + x^2 y, \\ \dot{y} = -y, \end{array} \right. \tag{5.1}$$

and

$$\left\{ \begin{array}{l} \dot{x} = -x + xy, \\ \dot{y} = -y. \end{array} \right. \tag{5.2}$$

System (5.1) has some particular solutions of the type $x(t) = \frac{2}{k}e^t$, $y(t) = ke^{-t}$ which let us to say that 0 is not a global attractor. The general solution of (5.2) is $x(t) = x_0 e^{-t-y_0(e^{-t}-1)}$, $y(t) = y_0 e^{-t}$ and hence 0 is a global attractor for system (5.2). One difference between systems (5.1) and (5.2) is the injectivity. While $F_1(x, y) = (-x + x^2 y, -y)$ is not injective, $F_2(x, y) = (-x + xy, -y)$ is injective.

In the sequel we discuss the effect of adding the injectivity condition to the condition of being a local attractor. Note that if F is an injective map then $\det DF(x)$ does not change sign, see Theorem 4. So, if 0 is a local hyperbolic attractor we get that $\det DF(x) \geq 0$ for all $x \in \mathbb{R}^2$. Observe that this last condition (with the strict inequality) together with the condition $\text{trace}(DF(x)) < 0$ are the MY conditions in the plane.

As we shall see, the injectivity condition is not enough to assure the global asymptotic stability. But it is so for some special systems.

Proposition 8

(i) *Consider the family of systems*

$$\begin{cases} \dot{x} = -x + f(x,y), \\ \dot{y} = -Ay, \end{cases} \tag{5.3}$$

where A is some positive real number and $f(x,y)$ smooth and beginning at least with second order terms. If the map $F(x,y) := (-x + f(x,y), -Ay)$ is injective then 0 is a global attractor of system (5.3).

(ii) *There are systems of the form*

$$\begin{cases} \dot{x} = -x + f(x,y), \\ \dot{y} = -Ay + g(x,y), \end{cases} \tag{5.4}$$

with A a some positive real number and $f(x,y)$ and $g(x,y)$ polynomials and beginning at least with second order terms, such that the map $(-x + f(x,y), -Ay + g(x,y))$ is injective and 0 is a local (but not global) attractor for system (5.4).

Proof. F injective implies that $\frac{\partial}{\partial x}(-x + f(x,y)) \le 0$, i. e. $-1 + \frac{\partial f}{\partial x}(x,y) \le 0$. The linear part of $F(x,y)$ is:

$$\begin{pmatrix} -1 + \frac{\partial f}{\partial x}(x,y) & * \\ 0 & -A \end{pmatrix}.$$

So, $\det(DF)(x,y) = A(1 - \frac{\partial f}{\partial x}(x,y)) \ge 0$ and $\operatorname{tr}(DF)(x,y) = -A - 1 + \frac{\partial f}{\partial x}(x,y) < 0$. Therefore the vector field is "almost" MY and injective. By using results of Olech's paper (see [20, Thm. 5]) we also can assert that 0 is a global attractor. This proves statement (i).

In order to prove (ii) consider the Liénard equation:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + \lambda y + y^2. \end{cases} \tag{5.5}$$

The origin is a critical point and its eigenvalues are

$$\mu_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

If we choose $\lambda < -2$ we get two real negative eigenvalues and so the origin is a local attractor for (5.5).

The phase portrait of system (5.5) is drawn in [15]. From that study it is clear that the origin is not a global attractor. On the other hand, trough the linear change which diagonalizes the matrix, system (5.5) becomes:

$$\begin{cases} \frac{du}{dt} = \mu_1 u + P(u,v), \\ \frac{dv}{dt} = \mu_2 v + Q(u,v), \end{cases}$$

for some polynomials P and Q. Trough the change of time $s = -\mu_1 t$ we get the system:

$$\begin{cases} \frac{du}{ds} = -u + f(u,v), \\ \frac{dv}{ds} = -\frac{\mu_2}{\mu_1} v + g(u,v). \end{cases} \tag{5.6}$$

Since systems (5.5) and (5.6) are topologically equivalents, they have the same phase-portrait. Hence system (5.6) provides an example which proves (ii). □

A way of interpreting the above proposition is saying that, for planar vector fields, the injectivity of a vector field is related to the existence of a global attractor just for "Markus-Yamabe" vector fields.

We end by observing that the counterexample given in [4] (see also Section 3.1) shows that there is no relation between injectivity and global asymptotic stability in $\mathbb{R}^n, n \geq 3$.

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