PICTURING PINCHUK'S PLANE POLYNOMIAL PAIR

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Abstract. Sergey Pinchuk discovered a class of pairs of real polynomials in two variables that have a nowhere vanishing Jacobian determinant and define maps of the real plane to itself that are not one-to-one. This paper describes the asymptotic behavior of one specific map in that class. The level of detail presented permits a good geometric visualization of the map. Errors in an earlier description of the image of the map are corrected (the complement of the image consists of two, not four, points). Techniques due to Ronen Peretz are used to verify the description of the asymptotic variety of the map.

1 Introduction

The strong real Jacobian conjecture stated that every polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with nowhere vanishing Jacobian determinant is univalent (one-to-one). This

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conjecture was refuted (for $n = 2$ and hence all larger n) in 1994 by Sergey Pinchuk, who provided a class of counterexamples [5]. One of the counterexamples is a map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ with $P(x, y)$ and $Q(x, y)$ polynomials of total degree 10 and 25, respectively [6]. That particular map is the primary focus of this paper. It can be described as follows.

Let $t = xy - 1, h = t(xt + 1), f = ((h + 1)/x)(xt + 1)^2, P = f + h, Q = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - (75/4)h^4$. Then $F(x, y) = (P(x, y), Q(x, y))$ is a real polynomial map from $\mathbb{R}^2$ to itself; its Jacobian determinant is everywhere positive; and it is not univalent.

This map has been considered elsewhere, in particular in [1], where the assertion was made that $F(\mathbb{R}^2)$ consists of all of $\mathbb{R}^2$ except for exactly four points. There were errors and oversights in the calculations, and two of the four points cited are actually in the image. In this paper, the complement of the image is identified as consisting of the points $(0, 0)$ and $(-1, -163/4)$ only.

The asymptotic behavior of the map is studied. In particular, the asymptotic variety of F (as defined by Ronen Peretz in [4]) is computed using Peretz's technique. Denote it by AV$[F]$. For the particular F studied here, AV$[F]$ admits a parameterization by two polynomials of degree two and five in a single variable.

The *asymptotic flower* of F (new terminology, see [2]) is the inverse image under F of AV$[F]$. Denote it by AF$[F]$. By construction, the restriction of F to a mapping from $\mathbb{R}^2 \setminus \mathrm{AF}[F]$ to $\mathbb{R}^2 \setminus \mathrm{AV}[F]$ is a proper map. For this particular F, it reduces to homeomorphisms of four simply connected domains in $\mathbb{R}^2$, each mapping onto one of the two simply connected components of $\mathbb{R}^2 \setminus \mathrm{AV}[F]$. This description provides a good geometric visualization of the map (and is supplemented by graphics).

2 Asymptotics of Pinchuk's Map

Pinchuk's map $F(x, y) = (P(x, y), Q(x, y))$ is most easily studied by considering the fibers $P = c$ of the map P, because P only has degree 10, whereas Q has degree 25. The following information and table are excerpted from [1]. The fiber $P = 0$ has five components and $P = -1$ has four components. In both cases ($c = 0$ and $c = -1$) the fibers can be computed and their components parameterized explicitly without great difficulty, because the polynomial $P - c$ factorizes simply. The other fibers are parameterized by the rational curve

$$x(h) = \frac{(c - h)(h + 1)}{(c - 2h - h^2)^2}$$

$$y(h) = \frac{(c - 2h - h^2)^2(c - h - h^2)}{(c - h)^2}$$

For a fixed value c, the components of the fiber $P = c$ are the images the map $h \mapsto (x(h), y(h))$ for values of h between successive poles (which occur when $h = c$ or $c - 2h - h^2 = 0$; no cancellation occurs as long as c is neither 0 nor -1). The table below summarizes the data on number of components and the range of Q for all fibers $P = c$. Q is always monotone (hence one-to-one) on any component of a fiber $P = c$, because the Jacobian determinant of P and Q is everywhere nonzero.

$P = c$	Ranges of Q on the components
$c > 0$	$(+\infty, q-)$, $(q-, q+)$, $(q+, -\infty)$, $(-\infty, +\infty)$
$c = 0$	$(0, 208)$, $(-\infty, 0)$, $(0, +\infty)$, $(-\infty, 0)$, $(208, +\infty)$
$-1 < c < 0$	$(+\infty, q-)$, $(q-, -\infty)$, $(-\infty, q+)$, $(q+, +\infty)$
$c = -1$	$(-\infty, -163/4)$, $(-\infty, -163/4)$, $(-163/4, +\infty)$, $(-163/4, +\infty)$
$c < -1$	$(+\infty, -\infty)$, $(-\infty, +\infty)$
Legend: (a, b) denotes the open interval from $\min(a, b)$ to $\max(a, b)$; $q+$ $(q-)$ = the value of Q at $h = -1 + \sqrt{1+c}$ (resp., $-1 - \sqrt{1+c}$);	

Table 1. Ranges of Q on fibers $P = c$ for Pinchuk's map

Remark 1 In [1] there was a typographical error in the formula for $x(h)$ (the term in the denominator was not squared). The computations leading to the results in Table 1 used the correct parameterization (the one shown above). Also, in [1] one of the points listed as not in the image of F was the point $(0, 208)$. However, a glance at the table shows that this cannot be correct; the value 208 lies in $(0, +\infty)$, which is the range of Q on one the components of the fiber $P = 0$.

Remark 2 In [1], the parameterization by $x(h), y(h)$ was introduced without any indication of how it arises. It comes from a straightforward process of solving the equations that define P, first for x and then for y. For example, if $P = c$ then the first step is $c = f + h$, then $c - h = ((h+1)/x)(xt+1)^2 = ((h+1)/x)(h/t)^2$, hence $xt^2 = (h+1)h^2/(c-h)$. From the defining equations again, $t = h - xt^2$, which allows solving for t in terms of h, then for $x = xt^2/t^2$, and finally for $y = (t+1)/x$.

The finite endpoints of ranges of Q occur precisely due to components of a fiber along which the x or y component blows up, but $Q(x, y)$ does not. Denote $x(h), y(h)$ by $x(c, h), y(c, h)$ to capture the dependence on c. Then one has the following rational identities

$$P(x(c,h), y(c,h)) = c$$

$$Q(x(c,h), y(c,h)) = \frac{1}{4(c-h)^2}\{197h^6 + (416 - 726c)h^5 + (252 - 1684c + 825c^2)h^4 + (-1224c + 2040c^2 - 300c^3)h^3 + (1648c^2 - 780c^3)h^2 + (-680c^3)h\}$$

The identities can be verified simply by substitution. For $c \neq 0, -1$ it can be checked that Q blows up when h tends to c (one of the poles of the parameterization), but not when h tends to either of the values $-1 + \sqrt{1+c}$, $-1 - \sqrt{1+c}$ (which are also poles of the parametrization, as they are the zeroes of $c - 2h - h^2$). The respective values of q are denoted by $q+$,$q-$. Of course, they depend on c. By definition, the asymptotic variety of a map [4] consists of points in the image plane that are limits of the images of points along a curve that tends to infinity in the original plane. By that definition, for each $c \neq 0, -1$ the points $(c, q+)$ and $(c, q-)$ are in the asymptotic variety, AV$[F]$, of the map F. These points can be obtained by simply substituting $c = 2h+h^2$ into the above rational identities for P and Q. To make life simple, u and v will be used as coordinates in the image plane, with x and y reserved for points in the original plane. Carrying out the indicated substitution yields the following parameterized curve in the image plane

$$u = P = c = 2h + h^2$$

$$v = Q = -(1/4)(1736h^3 + 1044h^2 + 1155h^4 + 300h^5)$$

The values of h that lead to $c = 0, -1$ are $h = -1$ $(c = 0)$, $h = 0$ $(c = 0)$, and $h = -2$ $(c = 0)$. The corresponding points (u, v) arising from the above parameterization are, respectively, $(-1, -163/4)$,$(0, 0)$, and $(0, 208)$. From Table 1, these points all belong to the AV$[F]$. So the entire curve lies in AV$[F]$. Using Peretz's technique of standard asymptotic identities, it will be shown below that this is, in fact, the entire asymptotic variety AV$[F]$. Figure 1 is a depiction of the variety.

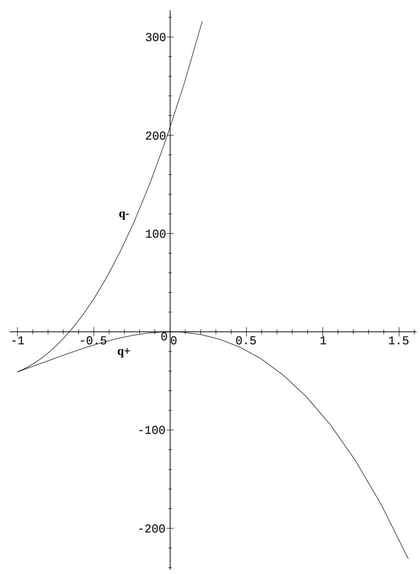

Figure 1. The asymptotic variety of Pinchuk's map.

The figure illustrates the fact that $q-$ is the larger of the two values of Q for a given c, whereas $q+$ is the smaller (except at $c = -1$, where they coincide). This can be seen easily by using a parametrization in terms of $w = h + 1$, for which the upper portion corresponds to $w < 0$ and the lower portion to $w > 0$.

Remark 3 In [1] it was claimed that there was a point of the form $u = c, v = d$ for some $-1 < c < 0$ which was not in the image of F, because the values of $q+$ and $q-$ supposedly coincided at that point. (That is, $d = q+ = q-$, hence d would not lie in any of the ranges of Q on the fiber $P = c$.) From the above figure, it is clear that there is no point where $q+$ or $q-$ coincide for $c > -1$.

Table 1 can be rewritten using the fact that $q+ < q-$ for $c \neq 0, -1$ to put all the intervals (a, b) that are ranges of Q on fibers of P in canonical form - that is, with $a < b$. The result is

$P = c$	Ranges of Q on the components
$c > 0$	$(-\infty, q+)$, $(q+, q-)$, $(q-, +\infty)$, $(-\infty, +\infty)$
$c = 0$	$(0, 208)$, $(-\infty, 0)$, $(0, +\infty)$, $(-\infty, 0)$, $(208, +\infty)$
$-1 < c < 0$	$(-\infty,\ q+)$, $(q+,\ +\infty)$, $(-\infty,\ q-)$, $(q-,\ +\infty)$
$c = -1$	$(-\infty, -163/4)$, $(-\infty, -163/4)$, $(-163/4, +\infty)$, $(-163/4, +\infty)$
$c < -1$	$(-\infty, +\infty)$, $(-\infty, +\infty)$

Table 2. Ranges of Q on fibers $P = c$ for Pinchuk's map - rewritten

This clearly shows that the only points omitted from the image of F are the points $(-1, -163/4)$ and $(0, 0)$.

3 The Peretz Method

This section uses the techniques described in Ronen Peretz's paper [4] to derive conditions that must be satisfied by any asymptotic values of the polynomial P. In the next section, those conditions will be used to show that AV$[F]$ is exactly the curve shown in Figure 1.

Observe first that the highest (total) degree term in P is x^6y^4, so P satisfies the Peretz normalization criterion $\deg(P) = \deg_x(P) + \deg_y(P)$. This implies that P has only x or y-finite asymptotic curves. In fact, P can only have asymptotic curves with $x \to \pm\infty$ and $y \to 0$, or vice versa. To search for y-finite asymptotic curves and the corresponding asymptotic values, follow the steps outlined by Peretz. That is, first write

$$P(x, y) = P_6x^6 + P_5x^5 + P_4x^4 + P_3x^3 + P_2x^2 + P_1x + P_0$$

where $P_0, \ldots, P_6$ are polynomials in y. This yields

$$\begin{aligned} P_6 &= y^4 \\ P_5 &= -4y^3 \\ P_4 &= 3y^3 + 6y^2 \\ P_3 &= -7y^2 - 4y \\ P_2 &= 3y^2 + 5y + 1 \\ P_1 &= -3y - 1 \\ P_0 &= y \end{aligned}$$

Then, assuming that P tends to the (finite) value C along an asymptotic curve (one that tends to infinity in the domain space), write down the Peretz assertions

$$\begin{aligned} &P_6x^6 + P_5x^5 + P_4x^4 + P_3x^3 + P_2x^2 + P_1x + P_0(0) \to C \\ &P_6x^5 + P_5x^4 + P_4x^3 + P_3x^2 + P_2x + P_1(0) \to 0 \\ &P_6x^4 + P_5x^3 + P_4x^2 + P_3x + P_2(0) \to 0 \\ &P_6x^3 + P_5x^2 + P_4x + P_3(0) \to 0 \\ &P_6x^2 + P_5x + P_4(0) \to 0 \\ &P_6x + P_5(0) \to 0 \\ &P_6(0) \to 0 \end{aligned}$$

These follow from the fact that if a product of two factors tends to a finite limit and one factor is x (which tends to $\pm\infty$), then the other factor tends to 0.

Look, from the bottom up, for the first assertion in which the term before $\to$ is not zero. This is the assertion containing $P_2(0)$. Write the assertion out in full. After judicious factorization it is

$$(yx)^4 - 4(yx)^3 + (3y + 6)(yx)^2 + (-7y - 4)(yx) + 1 \to 0$$

and since $y \to 0$, this implies that $xy \to r$, where r is a root of

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = (r - 1)^4$$

Remark 4 Justification. First, the power product xy must remain bounded, otherwise the expression would not approach a finite limit. Next, even if the polynomial has multiple distinct roots, the image of the curve must ultimately remain near a single root, otherwise the value of the expression would not be ultimately small (it would not tend to zero because of the trips from one root to another, necessarily involving points away from the roots). Similar reasoning applies later, when other power products than xy are considered.

Thus $xy \to 1$, which means that $(y - 1/x)x \to 0$. In other words

$$y = 1/x + o(1/x)$$

Next denote the error term by z, so that $y = 1/x + z$. Substitute $y = 1/x + z$ into the Peretz assertions to obtain the following ones

$$3z^2x^2 + x^6z^4 + 2z^2x^3 + 3z^3x^4 + 3zx \to C$$
$$z^4x^5 + 3z^3x^3 + 2z^2x^2 + 3z^2x + 6z + 3/x \to 0$$
$$z^4x^4 + 3z^3x^2 + 2z^2x - 5z - 4/x \to 0$$
$$x^3z^4 + (-3z^2 + z^2(3 + 3z))x - 2z + 2z(3 + 3z) + 3z^2 + (3 + 9z)/x + 3/x^2 \to 0$$
$$z^4x^2 - 6z^2 - 8z/x - 3/x^2 \to 0$$
$$z^4x + 4z^3 + 6z^2/x + 4z/x + 1/x^3 \to 0$$
$$0 \to 0$$

Using the facts that $1/x \to 0$, $z \to 0$, and $xz \to 0$, these assertions can be immediately simplified to

$$x^6z^4 + 2x^3z^2 + 3x^4z^3 \to C$$
$$x^5z^4 \to 0$$

plus five additional trivial assertions of the form $0 \to 0$. The fact that $x^5z^4 \to 0$ imposes the requirement that $z^4 = o(x^{-5})$. It follows that $z = o(|x|^{-5/4})$. No specific data on the form of the error term is implied. Finally, the assertion $(x^3z^2)^2 + (2 + 3xz)(x^3z^2) \to C$, together with $xz \to 0$, means that x^3z^2 tends to a root r of $r^2 + 2r - C = 0$. If $x^3z^2 \to r$ then $|z| = |r|^{1/2}|x|^{-3/2} + o(|x|^{-3/2})$. Since $5/4 < 3/2$, any such z automatically satisfies the $z = o(|x|^{-5/4})$ requirement.

To sum up, the following *necessary* requirements on the asymptotic behavior along a y-finite asymptotic curve for P with asymptotic limit C have been derived.

$$y = x^{-1} + s|x|^{-3/2} + o(|x|^{-3/2})$$

where $|s| = |r|^{1/2}$ and r is a root of $r^2 + 2r - C = 0$. If $r \neq 0$, then only one of the two possible choices of s occurs for a given asymptotic curve. However, either choice will lead to a curve with the right properties, since in either case $x^3z^2 \to r$.

To verify that these conditions suffice, denote again by z the (new) error term.

$$y = x^{-1} + s|x|^{-3/2} + z$$

Compute $P(x, x^{-1} + sx^{-3/2})$. The result is

$$s^4 + 2s^2 + (3s^3 + 3s)x^{-1/2} + (3s^2 + 1)x^{-1} + sx^{-3/2}$$

This is a correct formula for what must happen if $x \to +\infty$. If $x \to -\infty$ instead, $x^{-3/2}$ must be replaced by $|x|^{-3/2} = (-x)^{-3/2}$. Compute $P(x, x^{-1} + s(-x)^{-3/2})$. The result is

$$s^4 - 2s^2 + (-3s^2 + 3s)(-x)^{-1/2} + (3s^2 - 1)(-x)^{-1} + s(-x)^{-3/2}$$

To obtain the corresponding asymptotic identities in Peretz's standard form, substitute $1/x^2$ for x and y for s in the first, and $-1/x^2$ for x and y for s in the second, obtaining

$$P(1/x^2, yx^3 + x^2) = y^4 + 2y^2 + (3y^3 + 3y)x + (3y^2 + 1)x^2 + yx^3$$
$$P(-1/x^2, yx^3 - x^2) = y^4 - 2y^2 + (3y^3 - 3y)x + (3y^2 - 1)x^2 + yx^3$$

Remark 5 In [4] Peretz claims that to find all the asymptotic values of a polynomial P corresponding to y-finite asymptotic curves, it suffices to consider asymptotic identities of the form $P(1/x^k, yx^N + a_{N-1}x^{N-1} + \ldots + a_0) = a(x,y) \in \mathbb{R}[x,y]$. This appears to be an oversight. As this case shows, one must consider asymptotic identities involving $\pm 1/x^k$ when k is even, otherwise asymptotic values obtained as $x \to -\infty$ will be missed. It turns out that both P and Q satisfy asymptotic identities for each of the two asymptotic curves above. The (u,v) coordinates of points in AV$[F]$ are obtained by substituting $x = 0$ in the right hand sides of the asymptotic identities, and allowing y to vary. The right hand side of the first identity for P reduces to $y^4 + y^2$ for $x = 0$, so one can obtain only points with $u \geq 0$. In fact, one obtains the points in Figure 1 on the $q+$ portion of the curve, starting at $(0,0)$ and going to the right. The remaining points in AV$[F]$ all derive from the identities for the second asymptotic curve $(-1/x^2, yx^3 - x^2)$.

Next consider the case in which z, the error term, is not identically zero. As an illustration, compute $P(x, x^{-1} + sx^{-3/2} + z)$. The result is the same result obtained when $z = 0$ plus the following additional terms

$$\begin{aligned} z^4x^6 &+ 4sz^3x^{9/2} + 3z^3x^4 + (6s^2z^2 + 2z^2)x^3 \\ &+ 9sz^2x^{5/2} + 3z^2x^2 + (4s^3z + 4sz)x^{3/2} \\ &+ (3z + 9s^2z)x + 6szx^{1/2} + z \end{aligned}$$

Each of these terms tends to zero as a consequence of $z = o(|x|^{-3/2})$. So these are indeed all asymptotic curves, and the limiting asymptotic value obtained is independent of the form of z as long as $z = o(|x|^{-3/2})$. No new asymptotic limits are found. However, formally different asymptotic identities can be derived. For instance, from $y = x^{-1} + ax^{-3/2} + bx^{-2}$ one obtains the following asymptotic identity when $1/x^2$ is substituted for x and y is substituted for b

$$\begin{aligned} P(1/x^2, yx^4 + ax^3 + x^2) &= \\ x^4y^4 &+ (4x^3a + 3x^4)y^3 + (9x^3a + 6a^2x^2 + 3x^4 + 2x^2)y^2 \\ &+ (6x^3a + x^4 + 4ax + 4a^3x + 3x^2 + 9a^2x^2)y \\ &+ a^4 + 3ax + 3a^2x^2 + x^2 + x^3a + 3a^3x + 2a^2 \end{aligned}$$

Setting $x = 0$ in the right hand side to see what asymptotic limits are obtained yields $a^4 + 2a^2$, the same set of limit values as for the previous asymptotic identity. Note that the free parameter a yields the asymptotic values here, whereas all the y terms disappear if x is set equal to zero.

To look for x-finite asymptotic values, similar steps are taken, but there are fewer such steps since the powers of y extend only up to y^4. The first assertion, from the bottom up, that has a nonzero constant term on the left is (suitably rearranged)

$$(x^2y)^3 + (x^2y)^2(-4x+3) + (x^2y)(6x^2 - 7x + 3) + 1 \to 0$$

and as $x \to 0$ this implies that x^2y tends to a root r of the equation

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0$$

Thus the first approximation is

$$x = -y^{-1/2} + o(y^{-1/2})$$

with $y \to +\infty$ the only possibility. Substitute $x = -y^{-1/2} + z$ into the Peretz assertions to obtain

$$\begin{aligned} z^6y^4 &+ 18z^4y^3 - 4y^3z^5 + 6z^4y^2 - 47z^3y^2 + 36y^2z^2 + 8y \\ &- 4z^3y - 44zy + 41z^2y + 11 - 12z + 20z^4y^{(5/2)} + 12z^2y^{(1/2)} \\ &- 24zy^{(3/2)} - 34zy^{(1/2)} + 61z^2y^{(3/2)} - 24z^3y^{(3/2)} - 32z^3y^{(5/2)} \\ &- 6z^5y^{(7/2)} + 14y^{(1/2)} + 4y^{(-1/2)} \\ &\to 0 \end{aligned}$$

$$\begin{aligned} z^6y^3 &+ 18z^4y^2 - 4z^5y^2 + 6z^4y - 47z^3y + 36z^2y + 8 \\ &+ 36z^2 - 41z + 6y^{(-1)} + 20z^4y^{(3/2)} - 24z^3y^{(1/2)} - 24zy^{(1/2)} \\ &+ 61z^2y^{(1/2)} - 24zy^{(-1/2)} - 6z^5y^{(5/2)} - 32z^3y^{(3/2)} + 11y^{(-1)} \\ &\to 0 \end{aligned}$$

$$\begin{aligned} &- 18zy^{(-1/2)} + 40z^2y^{(-1/2)} - 20zy^{(-1)} + z^6y^2 - 6z^5y^{(3/2)} + 18z^4y \\ &- 32z^3y^{(1/2)} - 4yz^5 + 20z^4y^{(1/2)} + 4y^{(-3/2)} - 40z^3 + 33z^2 + 4y^{(-1)} \\ &\to 0 \end{aligned}$$

$$\begin{aligned} &z^6y - 6z^5y^{(1/2)} + 15z^4 - 20z^3y^{(-1/2)} + 15z^2y^{(-1)} - 6zy^{(-3/2)} + y^{(-2)} \\ &\to 0 \end{aligned}$$

and the trivial assertion $0 \to 0$. Every term containing a monomial of the form z^my^n tends to zero if $m \geq 2n$. The last two assertions collapse to the trivial $0 \to 0$. However, the next one from the bottom up reduces to $8 \to 0$. Since that cannot happen, it follows that there are no x-finite asymptotic limits.

4 The Asymptotic Variety of F

In the previous section it was shown that the asymptotic curves $(1/x^2, yx^3+x^2)$ and $(-1/x^2, yx^3-x^2)$, both defined for $x \neq 0$, are a basis for the asymptotic values of P, in the sense that every asymptotic value of P arises as a limit along one of these curves. These curves are also asymptotic curves that yield finite asymptotic values for Q. Specifically, one has the asymptotic identities

$$\begin{aligned} Q(1/x^2, yx^3+x^2) = &-(1/4)(1736y^6+1044y^4+1155y^8+300y^10) \\ &-(x/4)(5700y^7+6692y^5+2792y^3+1800y^9) \\ &-(x^2/4)(9636y^4+11250y^6+4500y^8+2432y^2) \\ &-(x^3/4)(6000y^7+680y+11100y^5+6140y^3) \\ &-(x^4/4)(4500y^6+1460y^2+5475y^4) \\ &-(x^5/4)(1800y^5+1080y^3)-75x^6y^4 \end{aligned}$$

$$\begin{aligned} Q(-1/x^2, yx^3-x^2) = &+(1/4)(1736y^6-1044y^4-1155y^8+300y^10) \\ &+(x/4)(-5700y^7+6692y^5-2792y^3+1800y^9) \\ &+(x^2/4)(9636y^4-11250y^6+4500y^8-2432y^2) \\ &+(x^3/4)(6000y^7-680y-11100y^5+6140y^3) \\ &+(x^4/4)(4500y^6+1460y^2-5475y^4) \\ &+(x^5/4)(1800y^5-1080y^3)+75x^6y^4 \end{aligned}$$

Substituting $x = 0$ to obtain the asymptotic values, both here and in the asymptotic identities for P, yields the following two parameterized curves that together make up the whole asymptotic variety

$$u = y^4+2y^2,\ v = -(1/4)(1736y^6+1044y^4+1155y^8+300y^{10})$$

$$u = y^4-2y^2,\ v = (1/4)(1736y^6-1044y^4-1155y^8+300y^{10})$$

These two parameterizations can be combined into one by putting $y^2 = h$ (for $h \geq 0$) in the first, and $y^2 = -h$ (for $h \leq 0$) in the second, which yields exactly the parameterization considered before

$$u = h^2+2h,\ v = -(1/4)(1736h^3+1044h^2+1155h^4+300h^5)$$

Remark 6 The functions t,h, and f introduced in the definition of P and Q all satisfy asymptotic identities with respect to each of the above two asymptotic curves. As Peretz remarked in [4, 3], examples such as Pinchuk's arise from finding pairs of polynomials with a nowhere vanishing Jacobian determinant in a real subalgebra of $\mathbb{R}[x,y]$ consisting of polynomials all of which have one or more shared asymptotic curves for which they satisfy asymptotic identities.

5 The Asymptotic Flower of F

In [2] the authors consider (primarily polynomial) maps of the real plane to itself that are proper. The *flower* of a map is the inverse image of the set of critical values (and a value is critical precisely if it is the image of a point at which the Jacobian determinant vanishes). Away from the flower the map is locally a covering map (proper and a local homeomorphism). In fact, it is a a covering map (over its image) on any connected component of the complement of the flower. In the case of Pinchuk's map the flower as defined above is empty, but the covering property fails to hold because the map is not proper. This suggests calling the above flower the *critical flower* and introducing as well the *asymptotic flower*, defined as the inverse image of the set of asymptotic values of the map. On the complement of the asymptotic flower, the restricted map to the complement of the set of asymptotic values is proper. This is because, by definition, as the asymptotic flower is approached, the image of a point will tend to infinity in the image plane or to an asymptotic value. But since the codomain of the restricted map is the complement of the set of asymptotic values, this means that the image is tending to infinity relative to that codomain. (Note. Asymptotic values can be defined in terms of limits of sequences as well on manifolds, using local pathwise connectedness to produce the appropriate curves. More general definitions are possible as well.) On each component of the complement of the *total flower* (the union of the critical and asymptotic flowers) the restricted map to the complement of the critical and asymptotic values will be a covering map over its image.

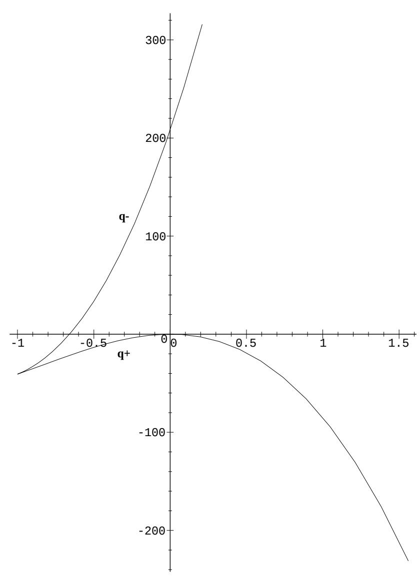

Figure 2. Three curves in the asymptotic variety.

Since the asymptotic variety AV[F] has been identified for the F at hand, it remains only to compute its inverse image to obtain AF[F], the asymptotic flower of F. By consulting Table 2, it becomes clear that

- exactly two points, $(0,0)$ and $(-1,-163/4)$, have no inverse images
- every other point of AV[F] has exactly one inverse image
- every point not in AV[F] has exactly two inverse images

This follows from a case by case check, cases corresponding to rows of the table. If one removes the two points that have no inverse images from AV[F], it breaks up into three connected curves. Call them $C1, C2, C3$, as follows.

$C1$ is the $q-$ curve, starting at $(-1,-163/4)$ and continuing up and to the right. $C2$ is the portion of the $q+$ curve starting at $(0,0)$ and continuing down and to the left, ending at $(-1,-163/4)$. Finally, $C3$ is the portion of the $q+$ curve ending at $(0,0)$ and arriving from down and to the right. Starting and ending points mentioned are not actually points of these curves, since they represent precisely points that were removed. The descriptions also imply orientations for the the three curves. Figure 2 shows the curves and their orientations.

Each point of each of the three curves has exactly one inverse image. Furthermore, as a starting or ending point, finite or infinite, is approached, the inverse image point tends to infinity. Thus the inverse image of each of $C1, C2, C3$ is a smooth curve in the plane (no singularities and no self-intersections) that tends to infinity at either end. Call these curves $D1$, $D2$, $D3$. By definition, AF[F] $= D1 \cup D2 \cup D3$, where the curves are considered as point sets.

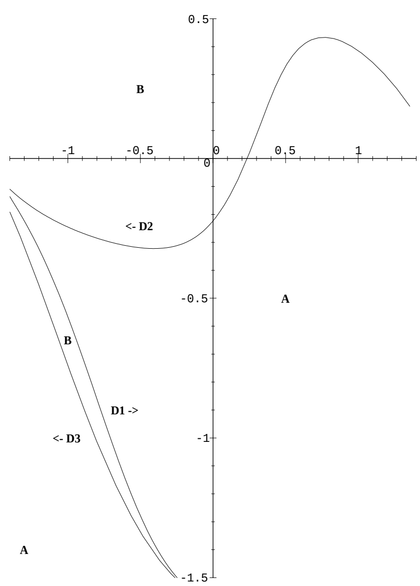

Figure 3. The asymptotic flower of Pinchuk's map.

Each of these curves $D1, D2, D3$ divides the plane into two simply connected parts (the Jordan curve Theorem), which may be described as the regions left and right of the curve, using the induced orientations to define left and right. Removing the curves $D1, D2, D3$ thus leaves exactly four simply connected open components. The restriction $F : \mathbb{R}^2 \setminus \mathrm{AF}[F] \to \mathbb{R}^2 \setminus \mathrm{AV}[F]$, maps each component into either the region L to the left of $\mathrm{AV}[F]$ or into the region R to its right. Each region mapped into L is, in fact, mapped homeomorphically onto L, because we are dealing with a covering of a simply connected region. Similarly for R. Label a connected component of $\mathbb{R}^2 \setminus \mathrm{AF}[F] = \mathbb{R}^2 \setminus (D1 \cup D2 \cup D3)$ with A if it maps to L, and with B if it maps to R. Two of the regions are labeled A, and two are labeled B. The global data of the map (the relations between the domains and curves) are best explained by a figure. Figure 3 depicts the component curves of $\mathrm{AF}[F]$ and their orientations, and also labels the regions defined by the curves. Figure 3 uses nonlinear scaling to produce a more comprehensible picture; the values plotted are actually the arctangents of the coordinates x and y. The figure was generated by solving for the inverse images of a large number of points on $\mathrm{AV}[F]$. The labeling of the regions can be checked by computing the images of a few points not in the flower (and can also be deduced to a large extent from the fact that F is orientation preserving, since its Jacobian determinant is everywhere positive).

References

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