Perfect matchings and Hamiltonian cycles in the preferential attachment model

Xavier Pérez-Giménez

joint work with

Alan Frieze[†], Paweł Prałat[*] and Benjamin Reiniger[‡]

AMS Sectional Meeting, University of St. Thomas, Minneapolis, October 2016

Some notation

n vertices, for $n \to \infty$

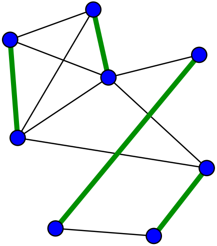
Definition

Event E_n holds **a.a.s.** (asymptotically almost surely) if

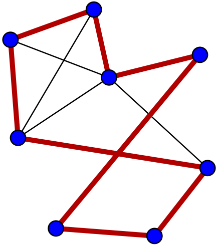
$$\lim_{n\to\infty} \mathbf{P}(E_n) = 1.$$

Perfect matchings and Hamilton cycles. . .

Perfect matching



Hamilton cycle



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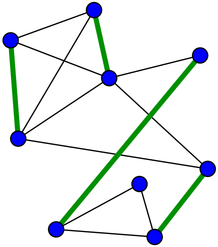

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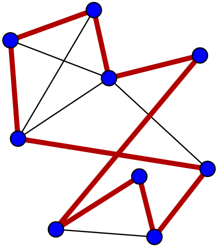

Perfect matching: matching of size $\lfloor n/2 \rfloor$

Perfect matchings and Hamilton cycles. . .

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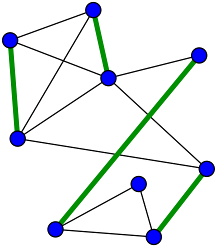

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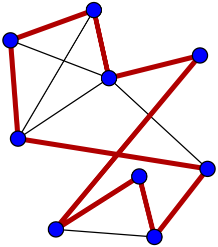

Perfect matching: matching of size $\lfloor n/2 \rfloor$

$\exists$ Hamilton cycle $\Longrightarrow$ $\exists$ Perfect matching

Is it true that $\begin{cases} \delta \geq 1 & \Rightarrow \exists \text{ perfect matching} \\ \delta \geq 2 & \Rightarrow \exists \text{ Hamilton cycle} \end{cases}$?

. . . in random graphs.

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Theorem (Bohman & Frieze '09):

Same for random m-out graphs.

$m = 3$ (out-degree)

n vertices

m-out model:

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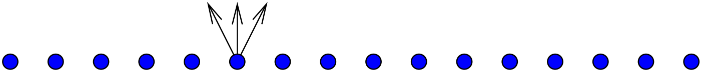

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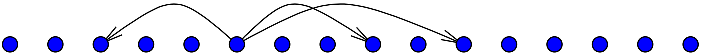

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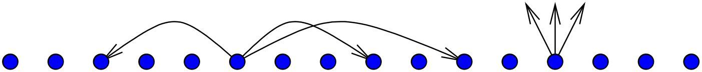

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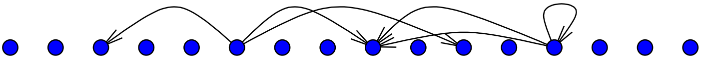

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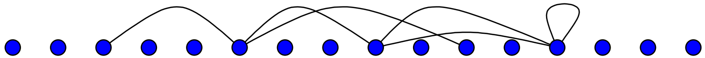

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Preferential attachment:

PA(n, m): Yule '25; Barabási & Albert '99

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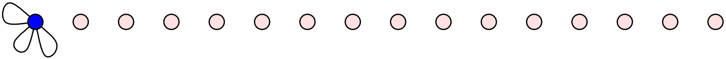

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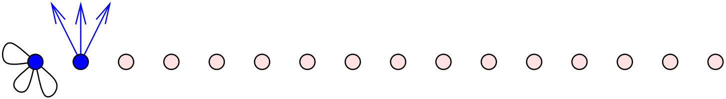

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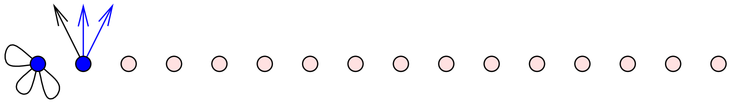

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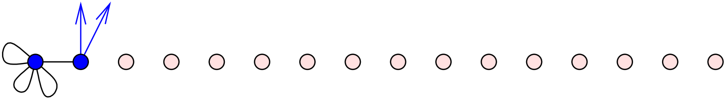

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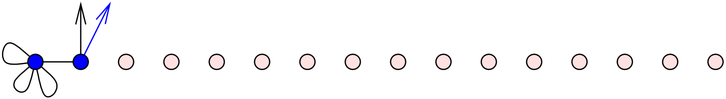

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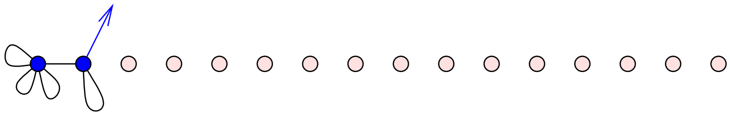

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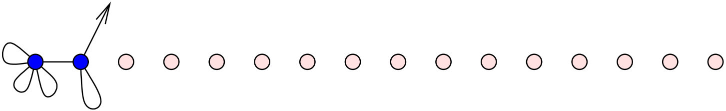

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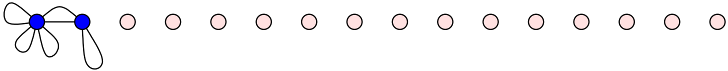

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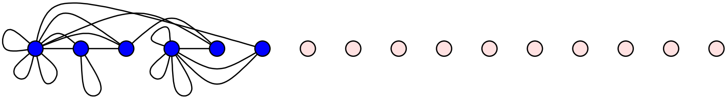

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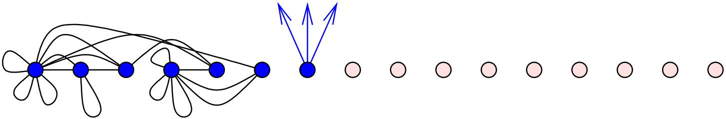

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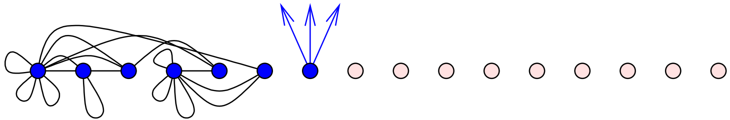

Preferential attachment rule: $\mathbf{P}(j \to i) \asymp \deg(i)$

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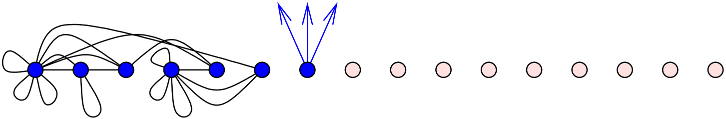

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Rich get richer!

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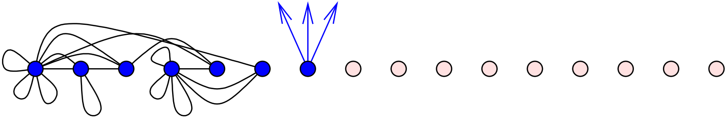

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Theorem (Bollobás, Riordan, Spencer & Tusnády '01):

For fixed $m \in \mathbb{N}$, a.a.s. PA(n, m) has a power-law degree distribution: For all $k \leq n^{1/15}$, $X_k \sim c_m k^{-2} n$, where X_k is the number of vertices of degree at least k.

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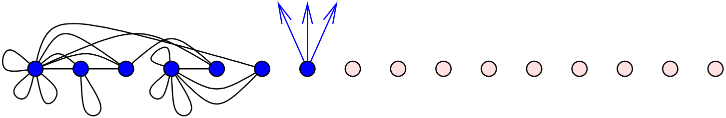

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Theorem (Bollobás & Riordan '04):

For $m \geq 2$, a.a.s. PA(n, m) is connected.

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UA(n, m): Intermediate model between m-out and PA(n, m)

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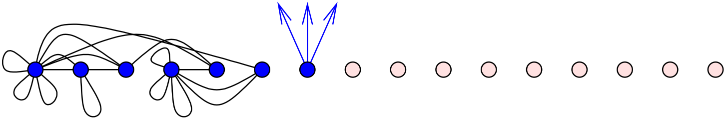

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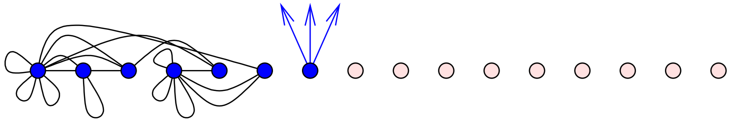

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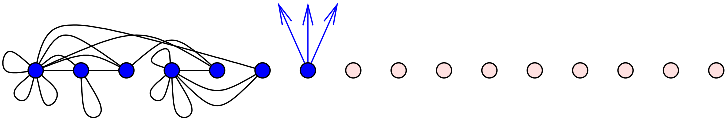

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Old get slightly richer!

Main results:

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- For $m \geq 159$, a.a.s. $\mathrm{UA}(n, m)$ has a perfect matching.
- For $m \geq 3{,}214$, a.a.s. $\mathrm{UA}(n, m)$ has a Hamilton cycle.
- For $m \geq 1{,}260$, a.a.s. $\mathrm{PA}(n, m)$ has a perfect matching.
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For $m = 2$, a.a.s. PA(n, m) has no Hamilton cycle.

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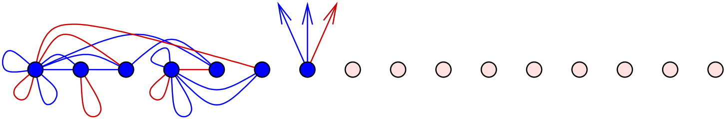
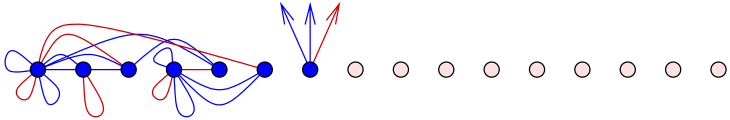
$\text{UA}(n, m), \quad m = m_1 + m_2$

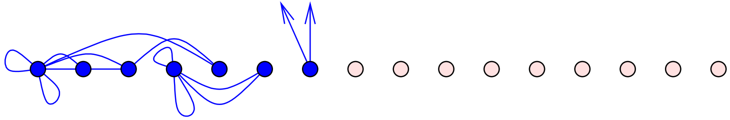


Two-round exposure:

UA(n, m), $m = m_1 + m_2$



UA(n, m_1)



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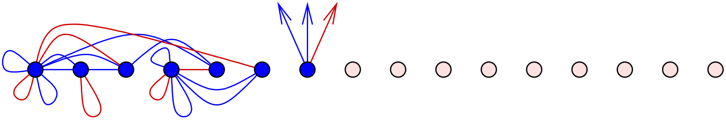

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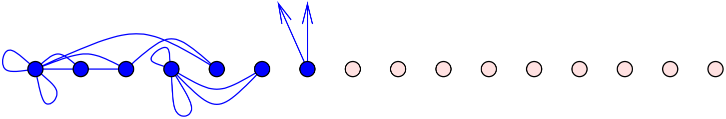

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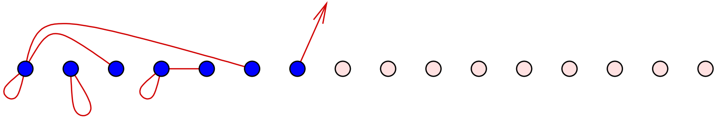

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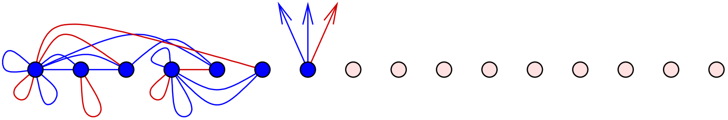

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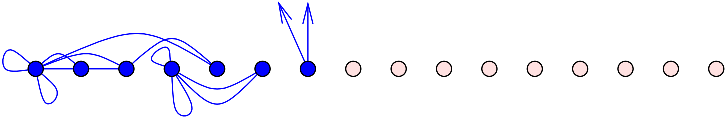

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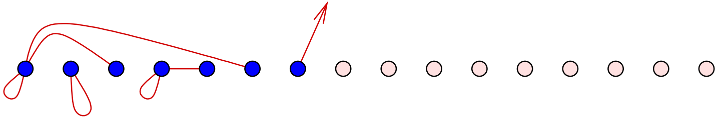

UA(n, m) = UA(n, m_1) ∪ UA(n, m_2) (independent)

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2-round exposure: $\mathrm{UA}(n, m) = \mathrm{UA}(n, m_1) \cup \mathrm{UA}(n, m_2)$

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First round — $\mathrm{UA}(n, m_1)$:

$\mathrm{UA}(n, m_1)$ a.a.s.:

- $\forall K$ s.t. $|K| \leq 2\epsilon n, \quad |N(K)| \geq 2|K| \quad$ (expansion),
- longest path has length $L \geq (1 - \epsilon/2)n$.

Still true if we add edges!

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Second round — add some edges of $\mathrm{UA}(n, m_2)$:

- We build sequence $\mathrm{UA}(n, m_1) = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{\epsilon n}$.
- G_i "improves" G_{i-1} if $L(G_i) > L(G_{i-1})$ or G_i contains HC.
- We show: for $1 \leq i \leq \epsilon n, \quad \mathbf{P}(G_i \text{ improves } G_{i-1}) \geq 3/4$.
- A.a.s. there are at least $(\epsilon/2)n$ improving steps, so we win!

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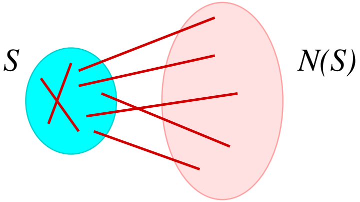
(Contains vertices not in S but adjacent to some vertex in S.)



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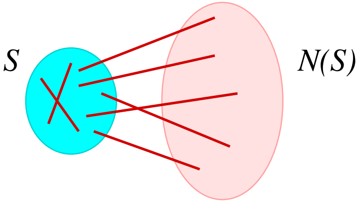

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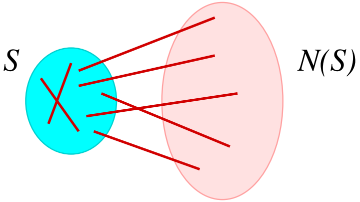

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We will use:

$|N(K)| \geq |K|$ for perfect matchings, and

$|N(K)| \geq 2|K|$ for Hamilton cycles.

Density properties

Lemma:

If $m = m(\epsilon)$ is large enough, then a.a.s.

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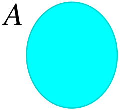

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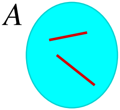

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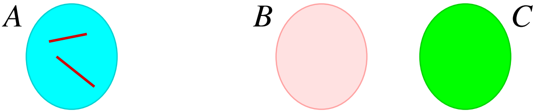

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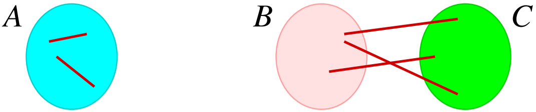

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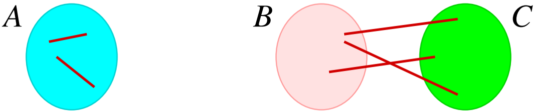

Proof (First moment method):

$\mathbf{P}(\exists$ independent sets of size $\epsilon n) \leq \mathbf{E}(\#$ of such sets$) = o(1)$.

Same for large pairs of sets with no edges across.

Density properties (continued)

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(i) $\Rightarrow \exists$ matching of size at least $(1 - \epsilon)n/2$.

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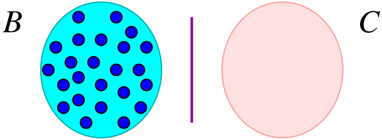

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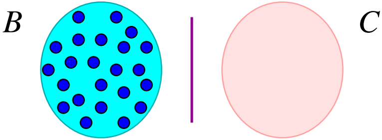

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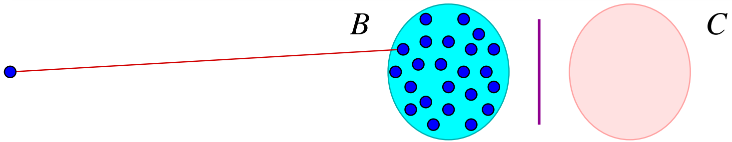

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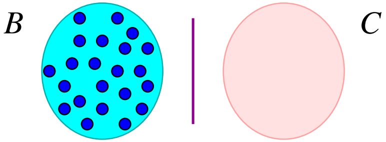

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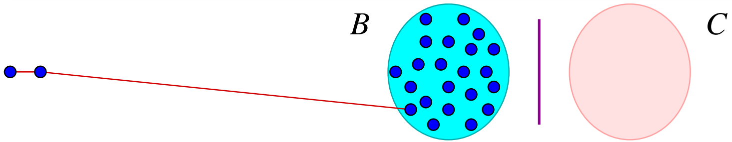

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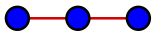

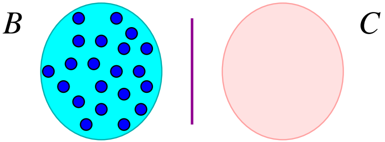

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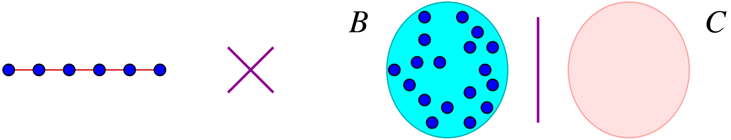

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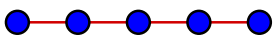

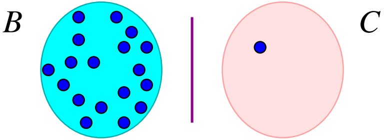

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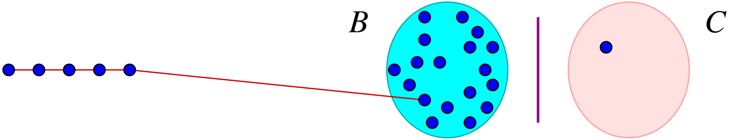

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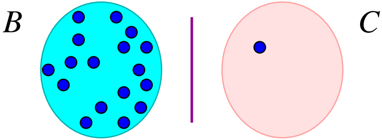

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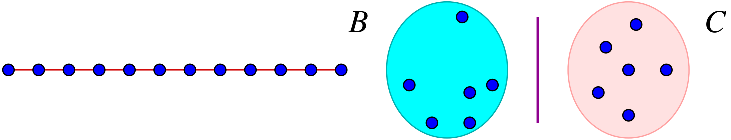

Building Hamilton cycles:

2-round exposure: $\mathrm{UA}(n, m) = \mathrm{UA}(n, m_1) \cup \mathrm{UA}(n, m_2)$

First round — $\mathrm{UA}(n, m_1)$:

$\mathrm{UA}(n, m_1)$ a.a.s.:

- $\forall K$ s.t. $|K| \leq 2\epsilon n, \quad |N(K)| \geq 2|K|$ (expansion),
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Still true if we add edges!

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Building Hamilton cycles: Longest paths

Definition:

Given graph G,
$A = \{v : v \text{ is an end of a longest path of } G\}$
Given $v \in A$,
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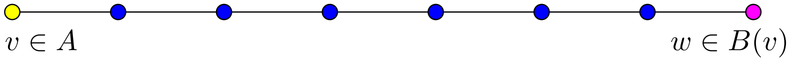

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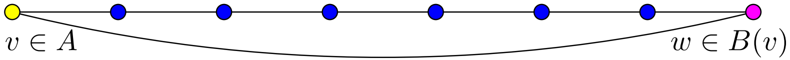

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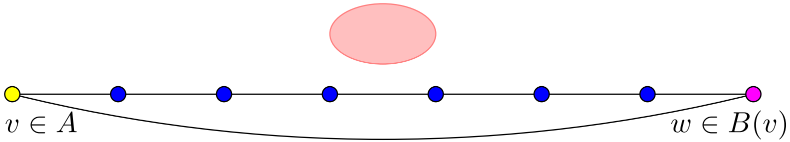

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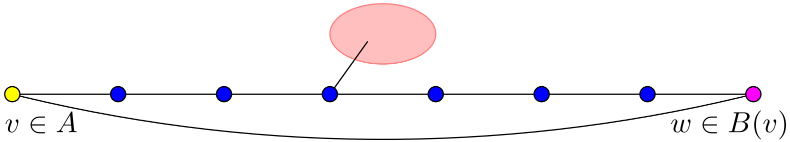

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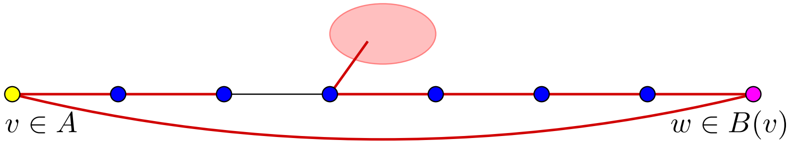

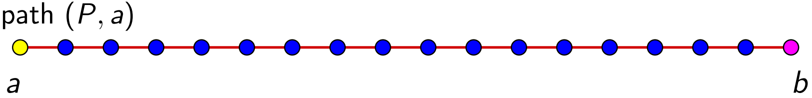
path (P, a)
a
b

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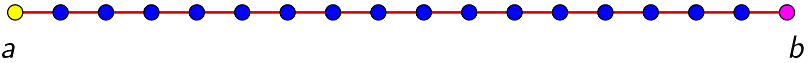

rotation $(P, a) \to (P', a)$

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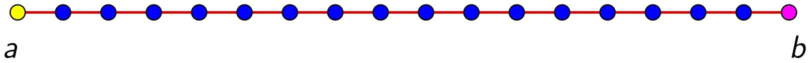

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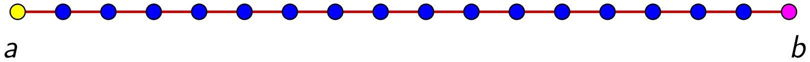

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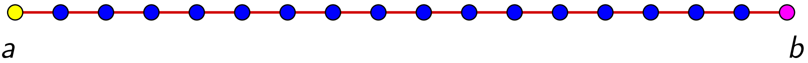

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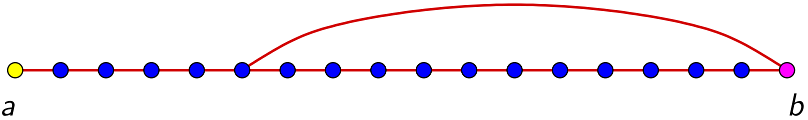
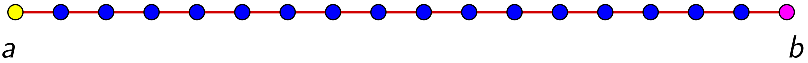


Building Hamilton cycles: Pósa's rotations

path (P, a)



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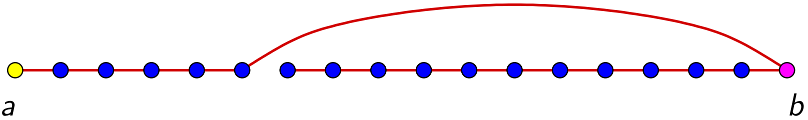

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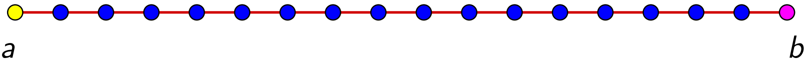

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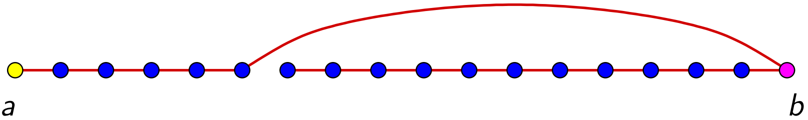

$\mathrm{END}(P, a) = \{$possible ends of (P, a) after sequence of rotations$\}$

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Lemma:

Let P be a longest path of a graph G and a one of its ends. Then,

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Proof:

- $N(\text{END}(P,a)) \subseteq V(P)$ (since P is a longest path)
- If $w \in N(\text{END}(P,a))$ then w is adjacent in P to some vertex in $\text{END}(P,a)$
- So $|N(\text{END}(P,a))| \leq 2|\text{END}(P,a))| - 1$.

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2-round exposure: $\mathrm{UA}(n, m) = \mathrm{UA}(n, m_1) \cup \mathrm{UA}(n, m_2)$

First round — $\mathrm{UA}(n, m_1)$:

$\mathrm{UA}(n, m_1)$ a.a.s.:

- $\forall K$ s.t. $|K| \leq 2\epsilon n, \quad |N(K)| \geq 2|K|$ (expansion),
- longest path has length $L \geq (1 - \epsilon/2)n$.

Still true if we add edges!

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Sequence $\text{UA}(n, m_1) = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{\epsilon n}$.
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Proof:
Let P be a longest path and v one of its ends.

- $|N(\mathrm{END}(P, v))| < 2|\mathrm{END}(P, v)|$,
- $\mathrm{END}(P, v) \subseteq B(v) \subseteq A$.
  So $|A| \geq |B(v)| \geq |\mathrm{END}(P, v)| > 2\epsilon n$.

Recall:

Sequence $\mathrm{UA}(n, m_1) = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{\epsilon n}$.
For each G_i: $\forall K$ s.t. $|K| \leq 2\epsilon n$, $|N(K)| \geq 2|K|$ (expansion).

Lemma:

For each G_i: $|A| \geq 2\epsilon n$ and $\forall v \in A$, $|B(v)| \geq 2\epsilon n$.

Proof:
Let P be a longest path and v one of its ends.

- $|N(\mathrm{END}(P, v))| < 2|\,\mathrm{END}(P, v)|$,
- $\mathrm{END}(P, v) \subseteq B(v) \subseteq A$.
  So $|A| \geq |B(v)| \geq |\,\mathrm{END}(P, v)| > 2\epsilon n$.

Recall:

Adding an edge between $v \in A$ and $w \in B(v)$ will improve G_i.

Building Hamilton cycles:

Construction of $\mathrm{UA}(n, m_1) = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{\epsilon n}$

For each $0 \leq i \leq \epsilon n - 1$:

- Consider G_i. (Update A, etc.)
- Pick youngest unmarked $v \in A$, and mark it.
- Expose edges in $\mathrm{UA}(n, m_1)$ from v to older vertices.

  Add them to G_i to form G_{i+1}
- If there are edges between v and $B(v)$, then G_{i+1} improves G_i.

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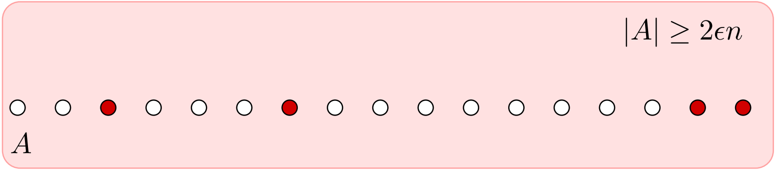

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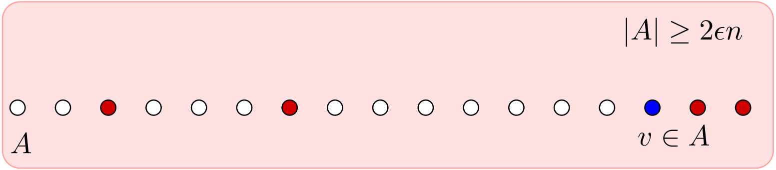

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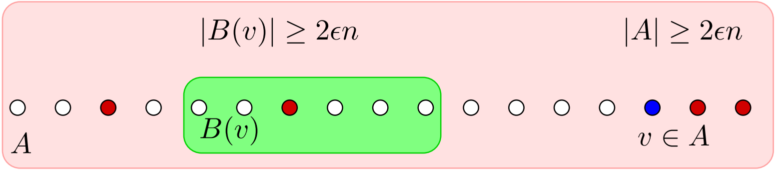

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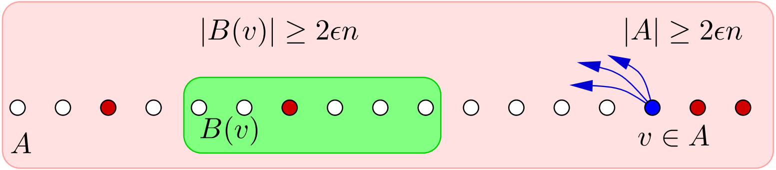

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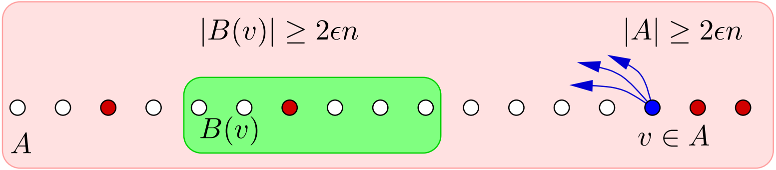

We have: $\mathbf{P}(G_{i+1} \text{ improves } G_i) \geq 1 - (1-\epsilon)^{m_2} > 3/4.$

Thank you