[1] (a) Give an example of a series which is only conditionally convergent. Justify your answer.

Answer: $1 + 1/2 - 1/3 + 1/4 \pm$ is convergent since it is an alternating series, and the absolute values of the terms are decreasing and have limit 0. But the absolute values of the terms give a p-series with $p = 1$ (i.e., the harmonic series), which is divergent.

(b) Determine whether or not the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!}$ is absolutely convergent. Justify your answer.

Answer: Using the ratio test, we have $|((-1)^{k+2}\frac{1}{(k+1)!})/((-1)^{k+1}\frac{1}{k!})| = 1/(k+1)$, which has limit 0, so the series $\sum_{k=1}^{\infty}\frac{1}{k!}$ converges, so $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!}$ is absolutely convergent.

[2] Using facts about alternating series, determine an $n \geq 0$ such that the difference between the sum S of the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!}$ and its partial sum $S_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k!}$ is less than 0.01. Justify your answer.

Answer: The difference is always at most the absolute value of the next term, which is $1/(n+1)!$, so it is enough to take n big enough so that $1/(n+1)! < 0.01$; i.e., $100 < (n+1)!$. Since $5! = 120$, we can take $n = 4$.

[3](a) Write down the Taylor polynomial $P_4(x)$ for $f(x) = e^x$.

Answer: $1 + x + x^2/2! + x^3/3! + x^4/4!$

(b) The series $\sum_{k=1}^{\infty}\frac{2^{k+1}}{k!}$ is obtained by evaluating the Taylor series of some function $g(x)$ at a particular value of x. Find $g(x)$ by modifying the Taylor series of e^x, and then find the sum S of the series exactly by evaluating $g(x)$ at an appropriate value of x. Explain your answer.

Answer: What we want is $2^2/1 + 2^3/2! + 2^4/3! + \cdots$. We know $e^2 = 1 + 2/1! + 2^2/2! + 2^3/3! + \cdots$, so $2(e^2 - 1) = 2(2 + 2^2/2! + 2^3/3! + \cdots) = 2^2/1 + 2^3/2! + 2^4/3! + \cdots$. I.e., the function is $g(x) = x(e^x - 1)$ (we could also use $g(x) = 2(e^x - 1)$), evaluated at $x = 2$, so the series sums to $2(e^2 - 1)$.

[4] Determine the radius and interval of convergence of the series $\sum_{k=1}^{\infty}\frac{(x-2)^k}{k3^k}$. Justify your answers, and make sure to explain what happens at both endpoints of the interval.

Answer: Using the ratio test, we take the limit, for $k \to \infty$, of $\left|\frac{\frac{(x-2)^{k+1}}{(k+1)3^{k+1}}}{\frac{(x-2)^k}{k3^k}}\right| = |(x-2)k/(3(k+1))|$. The limit is $|(x-2)/3|$, so except for the endpoints, the interval of convergence is $|(x-2)/3| < 1$, or $-1 < x < 5$, which tells us the radius of convergence is 3. When $x = -1$, the series is $\sum_{k=1}^{\infty}(-1)^k\frac{3^k}{k3^k}$, which converges by the alternating series test. When $x = 5$, the series is $\sum_{k=1}^{\infty}\frac{3^k}{k3^k}$, which is the harmonic series, and so diverges. Thus the interval of convergence is $-1 \leq x < 5$.

[5] Let C be the curve given parametrically by $x(t) = t^2$ and $y(t) = t^3 - t$.

(a) Determine the $x - y$ equation of the tangent line to the curve at the point $x = 4$, $y = 6$.

Answer: First, $x' = 2t$ and $y' = 3t^2 - 1$. The given point is given by $t = 2$, so $x'(2) = 4$ and $y'(2) = 11$. The slope of the tangent line is $y'(2)/x'(2) = 11/4$, so the equation (in point slope form) is $y - 6 = 11(x - 4)/4$.

(b) Find all values of t such that the curve has a horizontal tangent line.

Answer: I.e., solve $0 = y'(t)/x'(t) = (3t^2 - 1)/(2t)$; thus $3t^2 - 1 = 0$ so $t = \pm 1/\sqrt{3}$.

Harbourne's M107 Solutions Exam III Spring 2004

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